## Definition

A **transformation** $T$ from $R^n$ to $R^n$ is a mapping/function that associates each vector $\mathbf{v} = \langle v_1, \ldots, v_n \rangle$ with another vector $\mathbf{v}' = \langle v_1' \ldots, v_n' \rangle$. We write $T(\mathbf{v}) = \mathbf{v}'$, or

$$T : \begin{array}{c} v_1' = f_1(v_1, \ldots, v_n) \\ v_2' = f_2(v_1, \ldots, v_n) \\ \ldots \\ v_n' = f_n(v_1, \ldots, v_n) \end{array}$$

# Linear transformation

## Definition

A transformation $\mathbf{v}' = T(\mathbf{v})$ of $R^n$ to $R^n$ is **linear** if for every pair of vectors $\mathbf{u}$ and $\mathbf{v}$ in $R^n$, and every real number $c$, the following two conditions are satisfied:

$$\begin{aligned} T(\mathbf{u}+\mathbf{v}) &= T(\mathbf{u}) + T(\mathbf{v}) \\ T(c\mathbf{v}) &= c[T(\mathbf{v})] \end{aligned}$$

## Definition

A transformation $\mathbf{v}' = T(\mathbf{v})$ of $R^n$ to $R^n$ is **linear** if the components of $\mathbf{v}'$ are linear combinations of those of $\mathbf{v}$; that is, if

$$\begin{aligned} v_1' &= a_{11}v_1 + a_{12}v_2 + \cdots + a_{1n}v_n \\ v_2' &= a_{21}v_1 + a_{22}v_2 + \cdots + a_{2n}v_n \\ &\cdots \\ v_n' &= a_{n1}v_1 + a_{n2}v_2 + \cdots + a_{nn}v_n \end{aligned} \tag{1}$$

## Definition

Let $T$ be the linear transformation given by Equations (1). We define the **matrix associated with** $T$ as

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}.$$

With column vectors representing $\mathbf{v}$ and $\mathbf{v}'$,

$$\mathbf{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} \quad \text{and} \quad \mathbf{v}' = \begin{bmatrix} v_1' \\ \vdots \\ v_n' \end{bmatrix},$$

the linear transformation $T$ can be written in matrix form: $\mathbf{v}' = A\mathbf{v}$.

**Theorem**

*The $i^{th}$ column of the matrix associated with a linear transformation $T$ is the image of the vector whose components are all 0 except for a 1 in the $i^{th}$ position.*

## Definition

An **eigenvector** $\mathbf{v}$ of a linear transformation $T$ is a nonzero vector $\mathbf{v}$ that does not change direction when mapped by $T$, i.e., $T(\mathbf{v}) = \lambda\mathbf{v}$ for some constant $\lambda$.

The scalar $\lambda$ is called an **eigenvalue** of $T$; the nonzero vector $\mathbf{v}$ is called an eigenvector corresponding to the eigenvalue $\lambda$; together $(\lambda, \mathbf{v})$ is called an eigenpair.

**Theorem**

*Let $T$ be a linear transformation and $A$ be the matrix associated with $T$. Then $\lambda$ is an eigenvalue of $T$ if and only if it is a root of the* **characteristic equation** *of $A$:*

$$|A - \lambda I| = 0.$$

### Definition

A square matrix $A$ is said to be **symmetric** if $A^T = A$.

### Definition

In $R^n$, we define the **dot/inner product** of $\mathbf{u} = \langle u_1, u_2, \ldots, u_n \rangle$ and $\mathbf{v} = \langle v_1, v_2, \ldots, v_n \rangle$ as

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n.$$

Two nonzero vectors $\mathbf{u}$ and $\mathbf{v}$ are said to be **orthogonal** if $\mathbf{u} \cdot \mathbf{v} = 0$. Orthogonality and perpendicularity are the same in $R^2$ and $R^3$.

### Theorem

*If $A$ is a real, symmetric matrix, then:*

(1) *all eigenvalue of $A$ are real;*

(2) *eigenvectors corresponding to different eigenvalues are orthogonal.*