Vectors

Definition
Vectors are directed line segments.

Notation for vectors:

- by boldface type, e.g., $\mathbf{v}$;
- by the initial and terminal points, e.g., OP;
- by components, e.g., (x, y, z), $\langle x, y, z \rangle$, $\begin{bmatrix} x \\ y \\ x \end{bmatrix}$.

Definition

The *length* of a vector $\mathbf{v} = \langle v_x, v_y, v_z \rangle$ is

$$|\mathbf{v}| = \sqrt{v_x^2 + v_y^2 + v_z^2}.$$

The length of $\mathbf{v} = \langle v_1, \ldots, v_n \rangle$ is

$$|\mathbf{v}| = \sqrt{v_1^2 + \cdots + v_n^2}.$$

If $|\mathbf{v}| = 1$ then $\mathbf{v}$ is called a *unit vector*, denoted by $\hat{\mathbf{v}}$.

Let $\mathbf{u} = \langle u_1, \ldots, u_n \rangle$ and $\mathbf{v} = \langle v_1, \ldots, v_n \rangle$ be two vectors.

- Two vectors are equal if and only if they have the same length and direction, or if and only if they have the same components.
- Multiplication of a vector and a scalar: $\lambda\langle v_1, \ldots, v_n \rangle = \langle \lambda v_1, \ldots, \lambda v_n \rangle$.
- Addition or subtraction of two vectors: $\langle u_1, \ldots, u_n \rangle \pm \langle v_1, \ldots, v_n \rangle = \langle u_1 \pm v_1, \ldots, u_n \pm v_n \rangle$.

Definition : Scalar/Dot/Inner Product of Vectors

Let $\mathbf{u} = \langle u_1, \ldots, u_n \rangle$ and $\mathbf{v} = \langle v_1, \ldots, v_n \rangle$ be two vectors. The dot product of $\mathbf{u}$ and $\mathbf{v}$ is

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + \cdots + u_n v_n = \sum_{i=1}^{n} u_i v_i.$$

Properties of dot products:

- The dot product is **scalar** valued.
- $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$.
- $\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}| \cos\theta$, where θ is the angle between $\mathbf{u}$ and $\mathbf{v}$.
- $\mathbf{u} \cdot \mathbf{v} = 0$ if and only if $\mathbf{u}$ and $\mathbf{v}$ are orthogonal (in dimensions 2 and 3, we also say "perpendicular").
- $\cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}||\mathbf{v}|}$.

Definition : Vector/Cross/Outer Product

Let $\mathbf{u} = \langle u_x, u_y, u_z \rangle$ and $\mathbf{v} = \langle v_x, v_y, v_z \rangle$ be two vectors, $\vec{\mathbf{i}} = \langle 1, 0, 0 \rangle$, $\vec{\mathbf{j}} = \langle 0, 1, 0 \rangle$ and $\vec{\mathbf{k}} = \langle 0, 0, 1 \rangle$ be the *standard basis vectors*. Then

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \vec{\mathbf{i}} & \vec{\mathbf{j}} & \vec{\mathbf{k}} \\ u_x & u_y & u_z \\ v_x & v_y & v_z \end{vmatrix} = \begin{vmatrix} u_y & u_z \\ v_y & v_z \end{vmatrix} \vec{\mathbf{i}} - \begin{vmatrix} u_x & u_z \\ v_x & v_z \end{vmatrix} \vec{\mathbf{j}} + \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} \vec{\mathbf{k}}$$

or

$$\mathbf{u} \times \mathbf{v} = (u_y v_z - u_z v_y)\vec{\mathbf{i}} + (u_z v_x - u_x v_z)\vec{\mathbf{j}} + (u_x v_y - u_y v_x)\vec{\mathbf{k}}$$

Properties of cross product:

- The cross product is **vector** valued.
- $\mathbf{u} \times \mathbf{v}$ is perpendicular to both $\mathbf{u}$ and $\mathbf{v}$.
- $|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}||\mathbf{v}| \sin\theta$, where θ is the angle between $\mathbf{u}$ and $\mathbf{v}$.
- $\sin\theta = \frac{|\mathbf{u} \times \mathbf{v}|}{|\mathbf{u}||\mathbf{v}|}$

Theorem

The equation for the plane through the point (x_0, y_0, z_0) *perpendicular to the vector* $\langle A, B, C\rangle$ *is*

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0.$$

The vector $\langle A, B, C\rangle$ *is called a normal vector to the plane.*

Theorem

Let L be the line through the point (x_0, y_0, z_0) in direction $\mathbf{v} = \langle v_x, v_y, v_z \rangle$. Then the vector equation for L is:

$$\mathbf{r} = \mathbf{r}_0 + t\mathbf{v},$$

or

$$\langle x, y, z \rangle = \langle x_0, y_0, z_0 \rangle + t \langle v_x, v_y, v_z \rangle.$$

The parametric equations for L are

$$\begin{aligned} x =& x_0 + v_x t, \\ y =& y_0 + v_y t, \\ z =& z_0 + v_z t. \end{aligned}$$

If none of $v_x, v_y,$ and v_z is equal to zero, the line L has symmetric equations

$$\frac{x - x_0}{v_x} = \frac{y - y_0}{v_y} = \frac{z - z_0}{v_z}.$$