MATH 1210 Assignment #5 Solutions

Due: March 30, 2016; At the start of class

Reminder: all assignments *must* be accompanied by a signed copy of the honesty declaration available on the course website.

1. Given that the system has a unique solution, find it using the Cramer's rule:
   a)

$$\begin{aligned} 3x + 5y - 6z &= 8 \\ 7x - z &= 10 \\ 2x + 2y + z &= 2 \end{aligned}$$

   b)

$$\begin{aligned} 30x_1 + 40x_2 + 100x_3 - 143x_4 + x_5 &= 5 \\ 23x_1 - 80x_2 + 46x_3 - 127x_4 + 198x_5 &= -10 \\ 236x_1 + 24x_3 - 27x_4 + 80x_5 &= 0 \\ 123x_1 + 56x_2 - 34x_3 + 56x_5 &= 7 \\ 145x_1 - 64x_2 - 2x_3 + x_4 + 30x_5 &= -8 \end{aligned}$$

   Hint: For part (b), it may not be necessary to calculate 6 5×5 determinants. Look for simplifications!

Solution: a) Let A be the coefficient matrix of the system; A_i be A with the i-th column replaced by the vector of coefficients.

Then $|A| = \begin{vmatrix} 3 & 5 & -6 \\ 7 & 0 & -1 \\ 2 & 2 & 1 \end{vmatrix} = -7\begin{vmatrix} 5 & -6 \\ 2 & 1 \end{vmatrix} + \begin{vmatrix} 3 & 5 \\ 2 & 2 \end{vmatrix} = -7(5+12)+(6-10) = -123;$

$|A_1| = \begin{vmatrix} 8 & 5 & -6 \\ 10 & 0 & -1 \\ 2 & 2 & 1 \end{vmatrix} = 2\begin{vmatrix} 4 & 5 & -6 \\ 5 & 0 & -1 \\ 1 & 2 & 1 \end{vmatrix} = 2(-5\begin{vmatrix} 5 & -6 \\ 2 & 1 \end{vmatrix} + \begin{vmatrix} 4 & 5 \\ 1 & 2 \end{vmatrix}) = 2(-5(5+12)+(8-5)) = -164;$

$|A_2| = \begin{vmatrix} 3 & 8 & -6 \\ 7 & 10 & -1 \\ 2 & 2 & 1 \end{vmatrix} = 2\begin{vmatrix} 3 & 4 & -6 \\ 7 & 5 & -1 \\ 2 & 1 & 1 \end{vmatrix} = 2(\begin{vmatrix} 3 & 4 & -6 \\ 7 & 5 & -1 \\ 2+7 & 1+5 & 1-1 \end{vmatrix} = 2(\begin{vmatrix} 3 & 4 & -6 \\ 7 & 5 & -1 \\ 9 & 6 & 0 \end{vmatrix})$

$= 6(\begin{vmatrix} 3 & 4 & -6 \\ 7 & 5 & -1 \\ 3 & 2 & 0 \end{vmatrix}) = 6(3\begin{vmatrix} 4 & -6 \\ 5 & -1 \end{vmatrix} - 2\begin{vmatrix} 3 & -6 \\ 7 & -1 \end{vmatrix}) = 6(3(-4+30)-2(-3+42)) = 0;$

$|A_3| = \begin{vmatrix} 3 & 5 & 8 \\ 7 & 0 & 10 \\ 2 & 2 & 2 \end{vmatrix} = 2\begin{vmatrix} 3 & 5 & 4 \\ 7 & 0 & 5 \\ 2 & 2 & 1 \end{vmatrix} = 2(-7(5-8)-5(6-10)) = 2(21+20) = 82;$

and by Cramer's rule the solution is $x = \frac{|A_1|}{|A|} = \frac{-164}{-123} = \frac{4}{3}$, $y = \frac{|A_2|}{|A|} = 0$, $z = \frac{|A_3|}{|A|} = \frac{82}{-123} = -\frac{2}{3}$.

b) Let A be the coefficient matrix of the system; A_i be A with the i-th column replaced by the vector of coefficients. One can notice that the second column in the matrix of coefficients A is 8 times the vector of coefficients. This implies $|A_1| =$

$$= \begin{vmatrix} 5 & 40 & 100 & -143 & 1 \\ -10 & -80 & 46 & -127 & 198 \\ 0 & 0 & 24 & -27 & 80 \\ 7 & 56 & -34 & 56 & \\ -8 & -64 & -2 & 1 & 30 \end{vmatrix} = \begin{vmatrix} 5 & 40-8\cdot 5 & 100 & -143 & 1 \\ -10 & -80-8\cdot(-10) & 46 & -127 & 198 \\ 0 & 0-8\cdot 0 & 24 & -27 & 80 \\ 7 & 56-8\cdot 7 & -34 & 56 & \\ -8 & -64-8\cdot(-8) & -2 & 1 & 30 \end{vmatrix} =$$

$$= \begin{vmatrix} 5 & 0 & 100 & -143 & 1 \\ -10 & 0 & 46 & -127 & 198 \\ 0 & 0 & 24 & -27 & 80 \\ 7 & 0 & -34 & 56 & \\ -8 & 0 & -2 & 1 & 30 \end{vmatrix} = 0.$$ Similarly $|A_3| = 0, |A_4| = 0, |A_5| = 0$, and since it is given that the system has unique solution, $|A| \neq 0$ and $x_1 = \frac{|A_1|}{|A|} = 0, x_3 = \frac{|A_3|}{|A|} = 0, x_4 = \frac{|A_4|}{|A|} = 0, x_5 = \frac{|A_5|}{|A|} = 0$. Now $|A_2| =$

$$= \begin{vmatrix} 30 & 5 & 100 & -143 & 1 \\ 23 & -10 & 46 & -127 & 198 \\ 236 & 0 & 24 & -27 & 80 \\ 123 & 7 & -34 & 56 & \\ 145 & -8 & -2 & 1 & 30 \end{vmatrix} = \frac{1}{8}\begin{vmatrix} 30 & 40 & 100 & -143 & 1 \\ 23 & -80 & 46 & -127 & 198 \\ 236 & 0 & 24 & -27 & 80 \\ 123 & 56 & -34 & 56 & \\ 145 & -64 & -2 & 1 & 30 \end{vmatrix} = \frac{1}{8}|A|,$$

therefore $x_2 = \frac{|A_2|}{|A|} = \frac{\frac{1}{8}|A|}{|A|} = \frac{1}{8}$, so the unique solution is $(0, \frac{1}{8}, 0, 0, 0)$.

2. Let A and B be square matrices of the same size. Determine if the following statements are always true (justify your answer!):
a) If A is invertible, then AB is invertible;
b) If AB is invertible, then A is invertible.

Solution: a) Not true. For example, take $A = I$, $B = 0$; then A is invertible, but $AB = 0$ is not.
b) True. If AB is invertible, then $|AB| \neq 0$. If A was not invertible, $|A|$ would be equal to 0 and then $|AB| = |A||B| = 0|B| = 0$, which gives a contradiction. Therefore A is invertible.

3. Find all values of x for which the matrix $A = \begin{bmatrix} 2x & 5-x & 6 \\ x+3 & x-1 & 3x-3 \\ -40 & 10x^2+30 & 10x+80 \end{bmatrix}$ is singular.

Solution: The matrix A is singular if and only if $|A| = 0$.

$|A| = 10 \begin{vmatrix} 2x & 5-x & 6 \\ x+3 & x-1 & 3x-3 \\ -4 & x^2+3 & x+8 \end{vmatrix} = 10(2x \begin{vmatrix} x-1 & 3x-3 \\ x^2+3 & x+8 \end{vmatrix} -$ $(5-x) \begin{vmatrix} x+3 & 3x-3 \\ -4 & x+8 \end{vmatrix} + 6 \begin{vmatrix} x+3 & x-1 \\ -4 & x^2+3 \end{vmatrix}) = 10(2x(-3x^3+4x^2-2x+1) - (5-x)(x^2+23x+12) + 6(x^3+3x^2+7x+5) = -10(6x^4-15x^3-32x^2+59x+30)$.

Using the Rational Root theorem, we can find that two roots of the last polynomial are -2 and 3, and so $-10(6x^4-15x^3-32x^2+59x+30) = -10(x+2)(x-3)(6x^2-9x-5)$.

The remaining two roots are $\frac{9 \pm \sqrt{9^2+4\cdot 6\cdot 5}}{2\cdot 6} = \frac{9\pm\sqrt{201}}{12}$. Therefore A is singular for $x=-2$, 3, $\frac{9+\sqrt{201}}{12}$ or $\frac{9-\sqrt{201}}{12}$.

4. a) Using the adjoint method, find the inverse of $A = \begin{bmatrix} 3 & -4 & 5 \\ 6 & 7 & -1 \\ 2 & 8 & 1 \end{bmatrix}$;

b) Check by definition that the matrix found in (a) is indeed the inverse of A;

c) Use (a) to solve the system

$$\begin{aligned} 3x - 4y + 5z &= 13 \\ 6x + 7y - z &= 20 \\ 2x + 8y + z &= 23 \end{aligned}$$

d) Use (a) to solve the system

$$\begin{aligned} 3x + 6y + 2z &= 10 \\ -4x + 7y + 8z &= 5 \\ 5x - y + z &= 0 \end{aligned}$$

e) Solve the system $A^{-1} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

Solution:

a) Let's find cofactors of elements of A.

$c_{11} = \begin{vmatrix} 7 & -1 \\ 8 & 1 \end{vmatrix} = 15$; $c_{12} = -\begin{vmatrix} 6 & -1 \\ 2 & 1 \end{vmatrix} = -8$; $c_{13} = \begin{vmatrix} 6 & 7 \\ 2 & 8 \end{vmatrix} = 34$; $c_{21} = -\begin{vmatrix} -4 & 5 \\ 8 & 1 \end{vmatrix} = 44$; $c_{22} = \begin{vmatrix} 3 & 5 \\ 2 & 1 \end{vmatrix} = -7$; $c_{23} = -\begin{vmatrix} 3 & -4 \\ 2 & 8 \end{vmatrix} = -32$; $c_{31} = \begin{vmatrix} -4 & 5 \\ 7 & -1 \end{vmatrix} =$

-31; $c_{32} = -\begin{vmatrix} 3 & 5 \\ 6 & -1 \end{vmatrix} = 33$; $c_{33} = \begin{vmatrix} 3 & -4 \\ 6 & 7 \end{vmatrix} = 45$.

Now $|A| = a_{11}c_{11} + a_{12}c_{12} + a_{13}c_{13} = 3 \cdot 15 + (-4)(-8) + 5 \cdot 34 = 247$;

$$adjA = \begin{bmatrix} c_{11} & c_{21} & c_{31} \\ c_{12} & c_{22} & c_{32} \\ c_{13} & c_{23} & c_{33} \end{bmatrix} = \begin{bmatrix} 15 & 44 & -31 \\ -8 & -7 & 33 \\ 34 & -32 & 45 \end{bmatrix},$$

so $A^{-1} = \frac{1}{|A|} adjA = \frac{1}{247} \begin{bmatrix} 15 & 44 & -31 \\ -8 & -7 & 33 \\ 34 & -32 & 45 \end{bmatrix}$.

b) We need to check that $AA^{-1} = A^{-1}A = I$. Indeed, $AA^{-1} =$

$$\begin{bmatrix} 3 & -4 & 5 \\ 6 & 7 & -1 \\ 2 & 8 & 1 \end{bmatrix} \frac{1}{247} \begin{bmatrix} 15 & 44 & -31 \\ -8 & -7 & 33 \\ 34 & -32 & 45 \end{bmatrix} = \frac{1}{247} \begin{bmatrix} 3 & -4 & 5 \\ 6 & 7 & -1 \\ 2 & 8 & 1 \end{bmatrix} \begin{bmatrix} 15 & 44 & -31 \\ -8 & -7 & 33 \\ 34 & -32 & 45 \end{bmatrix} =$$

$$= \frac{1}{247} \begin{bmatrix} 247 & 0 & 0 \\ 0 & 247 & 0 \\ 0 & 0 & 247 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix};$$

$$A^{-1}A = \frac{1}{247} \begin{bmatrix} 15 & 44 & -31 \\ -8 & -7 & 33 \\ 34 & -32 & 45 \end{bmatrix} \begin{bmatrix} 3 & -4 & 5 \\ 6 & 7 & -1 \\ 2 & 8 & 1 \end{bmatrix} = \frac{1}{247} \begin{bmatrix} 247 & 0 & 0 \\ 0 & 247 & 0 \\ 0 & 0 & 247 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

c) The system can be rewritten as $A \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 13 \\ 20 \\ 23 \end{bmatrix}$, which has the unique solution

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = A^{-1} \begin{bmatrix} 13 \\ 20 \\ 23 \end{bmatrix} = \frac{1}{247} \begin{bmatrix} 15 & 44 & -31 \\ -8 & -7 & 33 \\ 34 & -32 & 45 \end{bmatrix} \begin{bmatrix} 13 \\ 20 \\ 23 \end{bmatrix} = \begin{bmatrix} \frac{362}{247} \\ \frac{515}{247} \\ \frac{837}{247} \end{bmatrix}.$$

d) The system can be rewritten as $A^T \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 10 \\ 5 \\ 0 \end{bmatrix}$, which has the unique solution

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = (A^T)^{-1} \begin{bmatrix} 10 \\ 5 \\ 0 \end{bmatrix} = (A^{-1})^T \cdot 5 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = \frac{5}{247} \begin{bmatrix} 15 & -8 & 34 \\ 44 & -7 & -32 \\ -31 & 33 & 45 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{110}{247} \\ \frac{405}{247} \\ -\frac{145}{247} \end{bmatrix}.$$

e) The system has the unique solution

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = (A^{-1})^{-1} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = A \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 & -4 & 5 \\ 6 & 7 & -1 \\ 2 & 8 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 12 \\ 11 \end{bmatrix}.$$

5. Let A be a square matrix such that $adjA = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 5 \\ -1 & -7 & -9 \end{bmatrix}$.

   a) Find $|adjA|$;

b) Find $|A|$;
c) Find A^{-1};
d) Find A.

Solution: a) $|adjA| = \begin{vmatrix} 1 & 2 & 3 \\ 1 & 3 & 5 \\ -1 & -7 & -9 \end{vmatrix} = \begin{vmatrix} 3 & 5 \\ -7 & -9 \end{vmatrix} - 2\begin{vmatrix} 1 & 5 \\ -1 & -9 \end{vmatrix} + 3\begin{vmatrix} 1 & 3 \\ -1 & -7 \end{vmatrix}$
$= 8 - 2(-4) + 3(-4) = 4$.
b) Since $A^{-1} = \frac{1}{|A|}adj(A)$, $adj(A) = |A| \cdot A^{-1}$ and $4 = |adj(A)| = |A|^3|A^{-1}| = |A|^3|A|^{-1} = |A|^2$. Therefore $|A| = \pm 2$.
c) $A^{-1} = \frac{1}{|A|}adj(A) = \pm\frac{1}{2}\begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 5 \\ -1 & -7 & -9 \end{bmatrix}$.
d) $A = (A^{-1})^{-1} = (\pm\frac{1}{2}\begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 5 \\ -1 & -7 & -9 \end{bmatrix})^{-1}$. We can use the adjoint method to find this inverse.
$det(\pm\frac{1}{2}\begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 5 \\ -1 & -7 & -9 \end{bmatrix}) = \pm\frac{1}{8}\begin{vmatrix} 1 & 2 & 3 \\ 1 & 3 & 5 \\ -1 & -7 & -9 \end{vmatrix} = \pm\frac{1}{8} \cdot 4 = \pm\frac{1}{2}$.
The cofactor matrix for it is $\frac{1}{4}\begin{bmatrix} 8 & 4 & -4 \\ -3 & -6 & 5 \\ 1 & -2 & 1 \end{bmatrix}$. So,
$A = \frac{1}{\pm\frac{1}{2}}\frac{1}{4}\begin{bmatrix} 8 & -3 & 1 \\ 4 & -6 & -2 \\ -4 & 5 & 1 \end{bmatrix} = \pm\frac{1}{2}\begin{bmatrix} 8 & -3 & 1 \\ 4 & -6 & -2 \\ -4 & 5 & 1 \end{bmatrix}$.

6. Use the direct method to find inverse of $\begin{bmatrix} 2\sqrt{3} & -1 & -3 \\ 2 & \sqrt{3} & 3\sqrt{3} \\ 0 & -2 & 2 \end{bmatrix}$.

Solution:

$$\left[\begin{array}{ccc|ccc} 2\sqrt{3} & -1 & -3 & 1 & 0 & 0 \\ 2 & \sqrt{3} & 3\sqrt{3} & 0 & 1 & 0 \\ 0 & -2 & 2 & 0 & 0 & 1 \end{array}\right] R_1 \leftrightarrow R_2 \left[\begin{array}{ccc|ccc} 2 & \sqrt{3} & 3\sqrt{3} & 0 & 1 & 0 \\ 2\sqrt{3} & -1 & -3 & 1 & 0 & 0 \\ 0 & -2 & 2 & 0 & 0 & 1 \end{array}\right] R_1 \leftarrow \frac{1}{2}R_1$$

$$\left[\begin{array}{ccc|ccc} 1 & \frac{\sqrt{3}}{2} & \frac{3\sqrt{3}}{2} & 0 & \frac{1}{2} & 0 \\ 2\sqrt{3} & -1 & -3 & 1 & 0 & 0 \\ 0 & -2 & 2 & 0 & 0 & 1 \end{array}\right] R_2 \leftarrow R_2 - 2\sqrt{3}R_1$$

$$\left[\begin{array}{ccc|ccc} 1 & \frac{\sqrt{3}}{2} & \frac{3\sqrt{3}}{2} & 0 & \frac{1}{2} & 0 \\ 0 & -4 & -12 & 1 & -\sqrt{3} & 0 \\ 0 & -2 & 2 & 0 & 0 & 1 \end{array}\right] R_2 \leftarrow -\frac{1}{4}R_2$$

$$\left[\begin{array}{ccc|ccc} 1 & \frac{\sqrt{3}}{2} & \frac{3\sqrt{3}}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 1 & 3 & -\frac{1}{4} & \frac{\sqrt{3}}{4} & 0 \\ 0 & -2 & 2 & 0 & 0 & 1 \end{array}\right] R_3 \leftarrow R_3 + 2R_2$$

$$\left[\begin{array}{ccc|ccc} 1 & \frac{\sqrt{3}}{2} & \frac{3\sqrt{3}}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 1 & 3 & -\frac{1}{4} & \frac{\sqrt{3}}{4} & 0 \\ 0 & 0 & 8 & -\frac{1}{2} & \frac{\sqrt{3}}{2} & 1 \end{array}\right] R_3 \leftarrow \frac{1}{8}R_3$$

$$\left[\begin{array}{ccc|ccc} 1 & \frac{\sqrt{3}}{2} & \frac{3\sqrt{3}}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 1 & 3 & -\frac{1}{4} & \frac{\sqrt{3}}{4} & 0 \\ 0 & 0 & 1 & -\frac{1}{16} & \frac{\sqrt{3}}{16} & \frac{1}{8} \end{array}\right] \begin{array}{l} R_1 \leftarrow R_1 - \frac{3\sqrt{3}}{2}R_3 \\ R_2 \leftarrow R_2 - 3R_3 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & \frac{\sqrt{3}}{2} & 0 & \frac{3\sqrt{3}}{32} & \frac{7}{32} & -\frac{3\sqrt{3}}{16} \\ 0 & 1 & 0 & -\frac{1}{16} & \frac{\sqrt{3}}{16} & -\frac{3}{8} \\ 0 & 0 & 1 & -\frac{1}{16} & \frac{\sqrt{3}}{16} & \frac{1}{8} \end{array}\right] R_1 \leftarrow R_1 - \frac{\sqrt{3}}{2}R_2 \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{\sqrt{3}}{8} & \frac{1}{8} & 0 \\ 0 & 1 & 0 & -\frac{1}{16} & \frac{\sqrt{3}}{16} & -\frac{3}{8} \\ 0 & 0 & 1 & -\frac{1}{16} & \frac{\sqrt{3}}{16} & \frac{1}{8} \end{array}\right].$$

Therefore, the inverse of $\begin{bmatrix} 2\sqrt{3} & -1 & -3 \\ 2 & \sqrt{3} & 3\sqrt{3} \\ 0 & -2 & 2 \end{bmatrix}$ is $\begin{bmatrix} \frac{\sqrt{3}}{8} & \frac{1}{8} & 0 \\ -\frac{1}{16} & \frac{\sqrt{3}}{16} & -\frac{3}{8} \\ -\frac{1}{16} & \frac{\sqrt{3}}{16} & \frac{1}{8} \end{bmatrix}$.

7. Determine if the following vectors are linearly independent or linearly dependent. Justify your answer.
a) $\langle 1, 2, 4\rangle, \langle -1, 2, 3\rangle, \langle 0, 3, -5\rangle$
b) $\langle 2, 3, -5, 8\rangle, \langle 3, 7, 9, 10\rangle, \langle -4, -6, 10, -16\rangle$
c) $\langle 23, 35, 57, 79\rangle, \langle 23, 34, 45, 56\rangle, \langle 87, 76, 65, 54\rangle, \langle 54, 43, 32, 32\rangle, \langle 35, 50, 75, 23\rangle$
d) $\langle -3, 2, 5, 4\rangle, \langle 3, 7, 8, -10\rangle, \langle 3, 5, 0, -9\rangle$
e) $\langle 1, 2\rangle, \langle 3, -7\rangle, \langle 8, 4\rangle$
f) $\langle 1, 0, 0, -3\rangle, \langle 0, 1, 2, 0\rangle, \langle 1, -3, 0, 0\rangle, \langle 4, 0, 6, -9\rangle$

Solution:

a) $\begin{vmatrix} 1 & -1 & 0 \\ 2 & 2 & 3 \\ 4 & 3 & -5 \end{vmatrix} = \begin{vmatrix} 2 & 3 \\ 3 & -5 \end{vmatrix} + \begin{vmatrix} 2 & 3 \\ 4 & -5 \end{vmatrix} = -19 - 22 = -41 \neq 0$, so the vectors are linearly independent.

b) The vectors are linearly dependent, because $2\langle 2,3,-5,8\rangle+\langle -4,-6,10,-16\rangle=0$.
c) The vectors are linearly dependent as $5>4$ vectors in 4-dimensional space.
d) Let $c_1\langle -3,2,5,4\rangle+c_2\langle 3,7,8,-10\rangle+c_3\langle 3,5,0,-9\rangle=0$.
The corresponding homogeneous system of linear equations for c_1,c_2,c_3 has the matrix

$$\begin{bmatrix}-3&3&3\\2&7&5\\5&8&0\\4&-10&-9\end{bmatrix} R_1\leftarrow -\frac{1}{3}R_1 \begin{bmatrix}1&-1&-1\\2&7&5\\5&8&0\\4&-10&9\end{bmatrix} \begin{matrix}R_2\leftarrow R_2-2R_1\\R_3\leftarrow R_3-5R_1\\R_4\leftarrow R_4-4R_1\end{matrix}$$

$$\begin{bmatrix}1&-1&-1\\0&9&7\\0&13&5\\0&-6&-5\end{bmatrix} \begin{matrix}R_3\leftarrow R_3+R_4\\R_3\leftarrow \frac{1}{7}R_3\end{matrix} \begin{bmatrix}1&-1&-1\\0&9&7\\0&1&0\\0&-6&-5\end{bmatrix} \begin{matrix}R_2\leftarrow R_2-9R_3\\R_4\leftarrow R_4+6R_3\end{matrix}$$

$$\begin{bmatrix}1&-1&-1\\0&0&7\\0&1&0\\0&0&-5\end{bmatrix} R_2\leftrightarrow R_3 \begin{bmatrix}1&-1&-1\\0&1&0\\0&0&7\\0&0&-5\end{bmatrix} \begin{matrix}R_3\leftarrow \frac{1}{7}R_3\\R_4\leftarrow R_4+5R_3\end{matrix} \begin{bmatrix}1&-1&-1\\0&1&0\\0&0&1\\0&0&0\end{bmatrix}$$

This implies that $c_2=c_3=0, c_1=c_2+c_3=0$, therefore the system has the unique solution $c_1=c_2=c_3=0$ and so the vectors are linearly independent.
e) The vectors are linearly dependent as $3>2$ vectors in 2-dimensional space.

f) $$\begin{vmatrix}1&0&1&4\\0&1&-3&0\\0&2&0&6\\-3&0&0&-9\end{vmatrix}=2(-3)\begin{vmatrix}1&0&1&4\\0&1&-3&0\\0&1&0&3\\-1&0&0&-3\end{vmatrix}=-6\begin{vmatrix}1&0&1&4\\0&1&-3&0\\0&1&0&3\\0&0&-1&-1\end{vmatrix}=$$

$$=-6\begin{vmatrix}1&-3&0\\1&0&3\\0&-1&-1\end{vmatrix}=-6\begin{vmatrix}1&-3&0\\0&3&3\\0&-1&-1\end{vmatrix}=-6\begin{vmatrix}1&-3&0\\0&0&0\\0&-1&-1\end{vmatrix}=0,$$ therefore the vectors are linearly dependent.

8. Let u,v,w be vectors in 3-dimensional space.
a) If $w=3u+2v$, express $v+w$ as a linear combination of u and w.
b) Prove that if u,v and w are linearly dependent, then $u-v,v$ and w are linearly dependent.
c) Is it true that if $u-v,v$ and w are linearly dependent, then u,v and w are linearly dependent? Justify your answer.
d) Is it true that if $u-v,v-w$ and $w-u$ are linearly dependent, then u,v and w are linearly dependent? Justify your answer.

Solution: a) Since $w=3u+2v$, $v=\frac{1}{2}w-\frac{3}{2}u$ and $v+w=\frac{1}{2}w-\frac{3}{2}u+w=-\frac{3}{2}u+\frac{3}{2}w$.
b) If u,v and w are linearly dependent, then there exist c_1,c_2,c_3, not all zeroes, such that $c_1u+c_2v+c_3w=0$. Then $0=c_1u+c_2v+c_3w=c_1u-c_1v+c_1v+c_2v+c_3w=$

$c_1(u - v) + (c_1 + c_2)v + c_3 w$, so 0 is expressed as a linear combination of $u - v, v$ and w. If all $c_1, c_1 + c_2$ and c_3 were zeroes, it would imply that $c_1 = c_2 = c_3 = 0$, which contradicts our assumption. Therefore $c_1, c_1 + c_2$ and c_3 are not all zeroes and $u - v, v$ and w are linearly dependent.

c) True. If $u - v, v$ and w are linearly dependent, then there exist c_1, c_2, c_3, not all zeroes, such that $c_1(u - v) + c_2 v + c_3 w = 0$. Then $0 = c_1(u - v) + c_2 v + c_3 w = c_1 u + (c_2 - c_1)v + c_3 w$, so 0 is expressed as a linear combination of u, v and w. If all $c_1, c_2 - c_1$ and c_3 were zeroes, it would imply that $c_1 = c_2 = c_3 = 0$, which contradicts our assumption. Therefore $c_1, c_2 - c_1$ and c_3 are not all zeroes and u, v and w are linearly dependent.

d) Not true. Vectors $u - v, v - w$ and $w - u$ are always linearly dependent, because their sum is equal to 0. So as a counterexample we can take any 3 linearly independent vectors, for example, $u = \langle 1, 0, 0 \rangle, v = \langle 0, 1, 0 \rangle, w = \langle 0, 1, 1 \rangle$.