Motivation for Linear Programming

The Tiny Toys Company produces two types of miniature toys, cars and robots. A miniature car will generate a profit of \$2.00 and each robot will generate a profit of \$2.50. Manufacturing these tiny toys involves two processes, namely printing and painting. The printing process takes 2 minute for a car, and only 1 minute for a robot. The painting process takes 4 minute for a car but 5 minutes for a robot. The Tiny Toys Company has at most 3 hours of printing and 10 hours of painting time available.

- How many toys of each of the two types should the company produce with the resources available in order to maximize its total profits?

Linear Programming Problem

Objective function Maximize or Minimize	

Constraints

printing	
painting	
other	
other	

Fundamental Theorem of Linear Programming

Theorem

It the linear programming problem has an optimal solution, then it must occur at a vertex (corner) of the feasible set.
Moreover, if the objective function has the same maximum (or minimum) value at two adjacent vertices of a feasible set, then it has the same maximum (or minimum) value at every point on the line segment joining these two vertices.

Method of Corners

1. Graph the feasible set and find its vertices.
2. Evaluate the objective function at each corner.
   (List the values in a counterclockwise, or clockwise, order.)
3. Select from the values in step 2 the optimum (maximum or minimum) value.
   (See the theorem for what to do when the optimum value occurs at two adjacent vertices.)

Example - Method of Corners

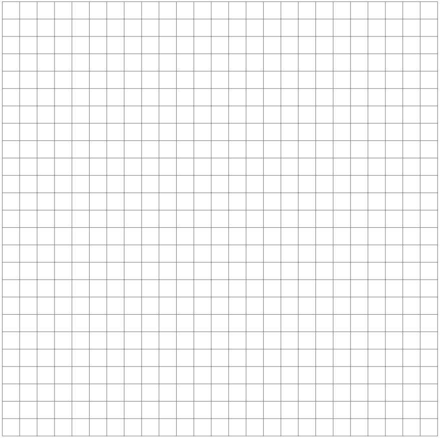
Maximize $P = 2x + \frac{5}{2}y$ subject to

$2x + y \leq 180$

$4x + 5y \leq 600$

$x \geq 0$

$y \geq 0$



Example - Method of Corners

Minimize $f = x - 2y$ subject to

$y - x \leq 1$

$3x - y \leq 6$

$x \geq 0$

$y \geq 0$

Example - Method of Corners

Minimize $f = 7x + 4y$ subject to

$2x + y \geq 7$

$x + 3y \geq 6$

$x \geq 0$

$y \geq 0$

Example - Method of Corners

Maximize $f = 2x - 5y$ subject to

$8x - 5y \geq -19$

$3x + 2y \leq 20$

$4x + 13y \geq -25$

Example - express as a linear programming problem

The Goodly Grains company produces two kinds of breakfast cereals made from wheat, oats and barley. *Wholesome Wholegrains* will sell for \$20 per kg bag, and *Healthy Hearty Heaps* will sell for \$25 per kg bag. *Wholesome Wholegrains* is 4 parts wheat, 1 part oat and 1 part barley. *Healthy Hearty Heaps* is 1 part wheat, 1 part oats and 2 parts barley. Goodly Grains currently has on hand 8000 kgs of wheat, 3500 kgs of oats, and 6000 kgs of barley.

- Assuming that Goodly Grains can sell all of the cereal it produces, then how much of each type of cereal should it produce in order to maximize its income from the current stock of grain.

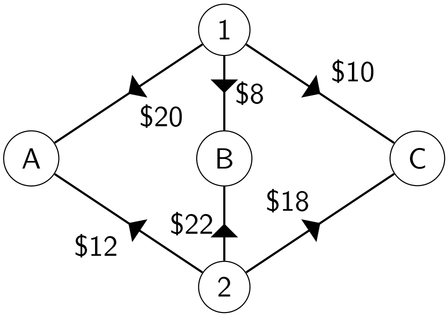
Example - express as a linear programming problem

Kelly Kirk is making beaded necklaces and bracelets to sell at a festival. She has 70 clasps and 1500 beads. Each necklace takes 30 beads and each bracelet takes 10 beads, both take a single clasp. Her display board has 20 squares, and each necklace takes two squares, each bracelet takes one.
Kelly sells each necklace for \$20 and each bracelet for \$10. She needs to begin the day with the display board filled, or she will not be able to set up her display booth.

- How many of each type of jewelry should Kelly make to maximize her income?

Example - Express as a linear programming problem

The Fishy-Fun company makes aquariums at two locations, Plant 1 and Plant 2. Plant 1 can manufacture 400 units, and Plant 2 can manufacture 600 units a month. They sell the aquariums at three different large retail locations, Store A, Store B and Store C. The minimum monthly stock requirements are 200, 300 and 400 units respectively.
The shipping costs per unit are outline in the following diagram:



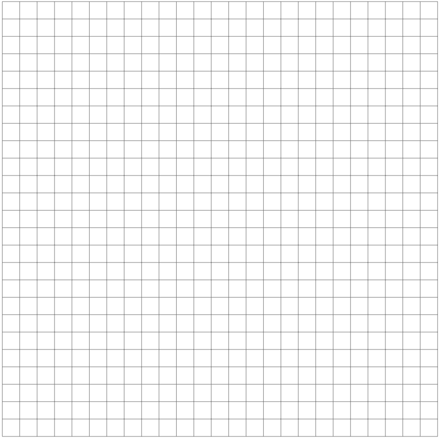
- What shipping schedule should the company adopt to meet the sales requirements while minimizing the shipping costs?

Linear Programming Application Example

John decides he will feed his pet poodle a combination of two dog foods. Each can of brand A contains 2 unit of protein, 2 unit of carbohydrates and 6 units of fat. Each can of brand B contains 6 unit of protein, 2 unit of carbohydrates and 2 units of fat. John wants his poodle to eat at least 12 units of protein, 8 units of carbohydrates and 12 units of fat each day.
Each can of brand A costs 80¢ and a can of brand B costs 50¢.

- How many cans of each brand of dog food should he feed to his pet every day?

Example - Linear Programming



Definition

Definition

A *matrix* is a rectangular array of values enclosed in parentheses.

We call the matrix an $m \times n$ matrix when this rectangular array has m rows and n columns. We refer to this value ($m \times n$) as the *dimension* (or size) of a matrix.

Matrix Examples

$$\begin{bmatrix} 1 \\ 5 \end{bmatrix} \qquad \begin{bmatrix} 2 & 4 \end{bmatrix} \qquad \begin{bmatrix} -1 & 3 \\ \frac{1}{5} & -7 \end{bmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{pmatrix}$$

General Matrix

$$A_{m\times n} = (a_{ij})_{m\times n} = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{pmatrix}$$

Example

If A is a 2×2 matrix such that:
$a_{11} = 2$
$a_{12} = \frac{1}{3}$
$a_{21} = x$
$a_{22} = 7$

Then $A = \begin{pmatrix} & \\ & \end{pmatrix}$

Row and Column Matrices

A matrix having only one row is referred to as a *row matrix*.

$$\begin{pmatrix} 1 & 3 & -7 & 2 & 5 \end{pmatrix}$$

A matrix having only one column is referred to as a *column matrix*.

$$\begin{pmatrix} 2 \\ 4 \\ -6 \\ 3 \\ 6 \end{pmatrix}$$

Square Matrices

A matrix where the number of rows is equal to the number of columns is called a *square* matrix. It is of size $n \times n$, and may be called *a square matrix of order n*.

$$\begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{2} & 1 & \frac{3}{2} & 2 \\ 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \end{bmatrix}$$

Diagonal Matrices

In a square matrix, the elements $a_{11}, a_{22}, a_{33}, \ldots, a_{nn}$ are called the *diagonal*.

A square matrix where all non zero elements are on the diagonal is called a *diagonal* matrix.

$$\begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Identity Matrix

A diagonal matrix having as diagonal elements equal to 1 is called an *identity* matrix. It is denoted I_n if it has dimensions $n \times n$.

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

symmetric matrix

A square matrix is said to be *symmetrical* if $a_{ij} = a_{ji}$.
This means that if you read off the n^{th} row it will be the same as the n^{th} column.

$$\begin{bmatrix} 1 & 3 \\ 3 & -2 \end{bmatrix} \qquad \begin{bmatrix} 1 & 2 & 3 \\ 2 & 7 & -5 \\ 3 & -5 & 4 \end{bmatrix} \qquad I_4 = \begin{bmatrix} 1 & -1 & 5 & -7 \\ -1 & 1 & 2 & 3 \\ 5 & 2 & 1 & 9 \\ -7 & 3 & 9 & 1 \end{bmatrix}$$

Zero Matrix

A matrix having all entries equal to zero called a *zero matrix*.

A zero matrix can be of any size $m \times n$ and is denoted $\widetilde{O}$ or $\widetilde{O}_{m\times n}$.

$$\widetilde{O}_{2\times 3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \widetilde{O}_{4\times 2} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \qquad \widetilde{O}_{3\times 3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Matrix Equality

Two matrices are equal if and only if they have the same dimensions and all of the same elements in the same places.

$$\begin{bmatrix} 1 & 3 \\ 3 & -2 \end{bmatrix} \neq \begin{bmatrix} 1 & 3 & 3 \\ 3 & -2 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 4 \\ 3 & -2 \end{bmatrix} \neq \begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 4 \\ 3 & -2 \end{bmatrix} \neq \begin{bmatrix} 4 & -2 \\ 1 & 3 \end{bmatrix}$$

$$\begin{pmatrix} 1 & 4 \\ 3 & -2 \end{pmatrix} = \begin{bmatrix} 1 & 4 \\ 3 & -2 \end{bmatrix}$$

Matrix operation - Scalar multiplication

We can multiply a matrix by a number, and we refer to the number as a *scalar*.

If A is an $n \times m$ matrix $A = (a_{ij})_{m\times n}$ and λ is a scalar then $\lambda A = (\lambda a_{ij})_{m\times n}$

$$3\begin{bmatrix} 1 & 2 & 3 \\ -1 & 7 & -5 \end{bmatrix} = \begin{bmatrix} 3(1) & 3(2) & 3(3) \\ 3(-1) & 3(7) & 3(-5) \end{bmatrix} = \begin{bmatrix} 3 & 6 & 9 \\ -3 & 21 & -15 \end{bmatrix}$$

Scalar multiplication - Examples

$$-2\begin{bmatrix} 1 & 4 \\ -3 & 6 \\ 2 & -7 \end{bmatrix}$$

$$11\begin{pmatrix} 3 & 1 & 2 \\ -4 & 5 & -5 \\ 2 & 7 & 9 \end{pmatrix}$$

$$0\begin{bmatrix} 1 & 4 & 1 \\ -3 & 6 & 1 \end{bmatrix}$$

Matrix Operation - Adding (or Subtracting)

We can add or subtract matrices only if they have the same dimensions. When they have the same dimension, we perform the addition or subtraction component-wise.

$$\begin{bmatrix} 1 & 2 & 3 \\ -1 & 7 & -5 \end{bmatrix} + \begin{bmatrix} 4 & 1 & 2 \\ 3 & -5 & 12 \end{bmatrix} = \begin{bmatrix} 1+4 & 2+1 & 3+2 \\ -1+3 & 7+(-5) & -5+12 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 3 & 5 \\ 2 & 2 & 7 \end{bmatrix}$$

Addition, Subtraction - Examples

$$\begin{bmatrix} 1 & 4 \\ -2 & 5 \\ 2 & -1 \end{bmatrix} + \begin{bmatrix} 2 & -4 \\ 3 & -2 \\ 2 & -7 \end{bmatrix}$$

$$\begin{pmatrix} 1 & 3 & 4 \\ -2 & 5 & -1 \\ 3 & 6 & 9 \end{pmatrix} + \begin{pmatrix} 3 & 1 & 3 \\ -2 & 4 & 0 \end{pmatrix}$$

$$\begin{bmatrix} 2 & 1 & -4 \\ 3 & -5 & 2 \end{bmatrix} - \begin{bmatrix} 1 & -3 & -2 \\ -3 & 6 & 1 \end{bmatrix}$$

Examples

$$3\begin{bmatrix} 2 & 0 \\ -1 & 4 \\ -1 & 2 \end{bmatrix} - 2\begin{bmatrix} 0 & 4 \\ 1 & -3 \\ 6 & -2 \end{bmatrix}$$

Find a matrix X that satisfies:

$$4X + 3\begin{pmatrix} 1 & 3 \\ 6 & -2 \end{pmatrix} = \begin{pmatrix} 7 & 13 \\ 6 & 10 \end{pmatrix}$$

Matrix Operation - Transpose

The *transpose* of a matrix A, which we denote A^T, is obtained by interchanging the rows and columns.
(The first row of A is the first column of A^T, the second row of A is the second column of A^T, etc.)

$$A = \begin{bmatrix} 3 & 1 \\ 4 & -2 \\ 7 & 12 \end{bmatrix} \qquad A^T = \begin{bmatrix} 3 & 4 & 7 \\ 1 & -2 & 12 \end{bmatrix}$$

Transpose - Examples

$$A = \begin{pmatrix} 3 & 2 & 1 \\ 4 & -5 & 7 \\ -3 & 2 & 6 \end{pmatrix} \qquad A^T =$$

$$B \begin{bmatrix} 2 & 1 & 4 & -5 \\ 0 & 3 & 7 & -1 \end{bmatrix} \qquad B^T =$$

Examples

$$A = \begin{bmatrix} 1 & -4 \\ 2 & 0 \\ 9 & 7 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 4 & 2 \\ 1 & 1 & -3 \end{bmatrix} \qquad C = \begin{bmatrix} 2 & 0 \\ -1 & 2 \end{bmatrix}$$

$$A + C^T =$$

$$A^T + B =$$

Matrix Operation - Matrix Multiplication

Matrix multiplication of type AB can only be performed if the number of columns of A is equal to the number of row of B.

$\underset{m\times n}{\text{A}}\,\underset{p\times q}{\text{B}} = \underset{m\times q}{\text{C}}$ can only be performed if $n = p$. Note the size of C.

$C_{m\times q} = (c_{ij})_{m\times q}$ where $c_{ij} = \sum_{k=1}^{n}(a_{ik})(b_{kj})$.

Matrix Multiplication Example

$$A = \begin{bmatrix} 1 & -4 \\ 2 & 0 \\ 3 & 7 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 4 \\ 2 & 1 \end{bmatrix}$$

$$AB =$$

Example

$$\begin{pmatrix} 2 \\ 4 \\ 5 \end{pmatrix} \begin{pmatrix} 1 & 3 & 6 \end{pmatrix} =$$

$$\begin{pmatrix} 1 & 3 & 6 \end{pmatrix} \begin{pmatrix} 2 \\ 4 \\ 5 \end{pmatrix} =$$

Example

$$2\begin{bmatrix}1\\-2\\3\end{bmatrix} =$$

$$\begin{bmatrix}2\end{bmatrix}\begin{bmatrix}1\\-2\\3\end{bmatrix} =$$

Example

$$A = \begin{pmatrix} 1 & -1 & 2 \\ 3 & 0 & -2 \end{pmatrix} \qquad B = \begin{pmatrix} 4 & 5 & 6 \\ 7 & 8 & 9 \\ 10 & 11 & 12 \end{pmatrix}$$

$$AB =$$

Example

$$A = \begin{pmatrix} 1 & 0 & -1 \\ 3 & 2 & -2 \\ 1 & 4 & -3 \end{pmatrix}$$

$$A^2 =$$

Example

$$A = \begin{pmatrix} 1 & 2 & -1 \\ 3 & 0 & 2 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & -1 \\ 2 & 4 \\ 3 & -1 \end{pmatrix}$$

$$AB =$$

$$BA =$$

Properties of Matrix Multiplication

$$A(BC) = (AB)C$$

$$\widetilde{0}A = \widetilde{0} \qquad\qquad A\widetilde{0} = \widetilde{0}$$

$$(AB)^T = B^T A^T$$

$$A_{m\times n} I_n = I_m A_{m\times n} = A_{m\times n}$$

$$A(B + C) = AB + AC \qquad\qquad (A + B)C = AC + BC$$

Systems of Linear Equations

When we consider solving a system of two or more equations with two or more variables, we are solving a system of linear equations.

Example

$$\begin{array}{rcrcrl} x & + & y & + & 2z = & 8 \\ -x & - & 2y & + & 3z = & 1 \\ 3x & - & 7y & + & 4z = & 10 \end{array}$$

Definiton

Definition

When a system of linear equations has one or more solutions, the system is said to be *consistent*. When the system has no solutions, it is said to be *inconsistent*.

Examples

$$\begin{aligned} x &+ y = 4 \\ 2x &- 2y = 4 \end{aligned}$$

$$\begin{aligned} x &- y = 4 \\ 2x &- 2y = 4 \end{aligned}$$

$$\begin{aligned} x &- y = 2 \\ 2x &- 2y = 4 \end{aligned}$$

Example

$$\begin{aligned} x &+ y + 2z = 8 \\ -x &- 2y + 3z = 1 \\ 3x &- 7y + 4z = 10 \end{aligned}$$

Elementary Operations

1. Interchange the order of equations.

2. Multiply an equation by a nonzero number.

3. Add a (nonzero) multiple of one equation to another.

Theorem: If elementary operations are performed on a system of linear equations, the resulting system is equivalent to the original system; they have the same solutions.

Augmented Matrix

We transform a system of equations into an *augmented matrix* by striping off the relevant coefficients and arranging into a matrix, along with the constants.

Example:

$$\begin{array}{rcrcrl} x & + & y & + & 2z = & 8 \\ -x & - & 2y & + & 3z = & 1 \\ 3x & - & 7y & + & 4z = & 10 \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 2 & 8 \\ -1 & -2 & 3 & 1 \\ 3 & -7 & 4 & 10 \end{array}\right)$$

Elementary Row Operations

1. Interchange rows. $R_i \leftrightarrow R_j$
2. Multiply a row by a nonzero number. $R_i \to aR_i$
3. Add a (nonzero) multiple of one row to another row.
   $R_i \to R_i + bR_j$

Example

$$\left(\begin{array}{ccc|c} 1 & 1 & 2 & 8 \\ -1 & -2 & 3 & 1 \\ 3 & -7 & 4 & 10 \end{array}\right)$$

Row Echelon Form

An Augmented matrix is said to be in *row echelon form* (REF) if it satisfies the following:

1. The leftmost nonzero entry in every row is a 1 (called a *leading* 1).
2. The leading 1 for each row is to the left of leading 1 in every row below it.
3. Every entry below a leading 1 is a zero.
4. Any rows of all 0's are below all rows that have leading 1's.

Examples - Is the matrix in REF?

$$\left(\begin{array}{cc|c} 1 & 2 & 4 \\ 0 & 1 & 3 \end{array}\right) \qquad \left(\begin{array}{ccc|c} 1 & -1 & 2 & 5 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 3 & 7 \end{array}\right) \qquad \left(\begin{array}{ccc|c} 1 & 0 & 4 & 6 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\left(\begin{array}{cc|c} 1 & 2 & 4 \\ 0 & 1 & 3 \\ 0 & 1 & 5 \end{array}\right) \qquad \left(\begin{array}{ccc|c} 1 & 1 & 2 & 8 \\ 0 & 0 & 1 & -9 \\ 0 & 0 & 0 & 0 \end{array}\right) \qquad \left(\begin{array}{ccc|c} 1 & 3 & 2 & 5 \\ 0 & 2 & 4 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\left(\begin{array}{cccc|c} 1 & 2 & 4 & 1 & 1 \\ 0 & 0 & 1 & 1 & 2 \\ 0 & 1 & 5 & 1 & 3 \\ 0 & 0 & 0 & 1 & 4 \end{array}\right) \qquad \left(\begin{array}{cccc|c} 1 & 1 & 2 & 8 & 1 \\ 0 & 0 & 1 & -9 & 2 \\ 0 & 0 & 0 & 0 & 3 \end{array}\right) \qquad \left(\begin{array}{cc|c} 1 & 3 & 2 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \end{array}\right)$$

Gaussian Elimination

The following step by step procedure will take an augmented matrix and reduce it to REF using elementary row operations.

1. Find the leftmost variable column that has a least one nonzero entry in it. Use entries in this column to create a 1 in the top row (a leading 1 for this row).
2. Add multiples of the top row to the rows beneath it to create 0's beneath the leading 1.
3. Pretend the top row is not there and repeat steps 1 and 2 on the remaining rows.

To solve a system of equations using Gaussian elimination, we find the REF of the augmented matrix, and then perform back substitution on the equations from the REF form.

Example - Gaussian elimination

$$\begin{array}{rcrcrcr} x & + & 2y & - & z & = & 6 \\ 2x & - & y & + & 3z & = & -3 \\ x & + & 3y & + & 2z & = & 5 \end{array}$$

Example - Gaussian elimination

$$\begin{array}{rcrcrcr} x & - & 2y & + & z & = & 7 \\ 3x & + & 4y & + & 3z & = & 1 \\ 2x & + & y & + & 2z & = & 2 \end{array}$$

Example - Gaussian elimination

$$
\begin{array}{rcrcrcrcr}
x_1 & + & 2x_2 & - & x_3 & + & x_4 & = & 10 \\
2x_1 & + & 5x_2 & - & 3x_3 & + & 2x_4 & = & 24 \\
-x_1 & + & x_2 & + & 3x_3 & + & 9x_4 & = & 17
\end{array}
$$

Reduced Row Echelon Form

An Augmented matrix is said to be in *row echelon form* (RREF) if it satisfies the following:

1. The leftmost nonzero entry in every row is a 1 (called a *leading* 1).
2. The leading 1 for each row is to the left of leading 1 in every row below it.
3. Every entry above and below a leading 1 is a zero.
4. Any rows of all 0's are below all rows that have leading 1's.

Examples - Is the matrix in RREF?

$$\left(\begin{array}{cc|c} 1 & 0 & 4 \\ 0 & 1 & 3 \end{array}\right) \qquad \left(\begin{array}{ccc|c} 1 & 0 & 4 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 6 & 7 \end{array}\right) \qquad \left(\begin{array}{ccc|c} 1 & 0 & -1 & 5 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 5 \end{array}\right) \qquad \left(\begin{array}{ccc|c} 1 & 5 & 0 & 8 \\ 0 & 0 & 1 & -9 \\ 0 & 0 & 0 & 0 \end{array}\right) \qquad \left(\begin{array}{ccc|c} 1 & 0 & 2 & 5 \\ 0 & 2 & 4 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 1 & 0 & 0 & 3 \\ 0 & 0 & 0 & 1 & 4 \end{array}\right) \qquad \left(\begin{array}{cccc|c} 1 & 1 & 2 & 8 & 1 \\ 0 & 0 & 1 & -9 & 2 \\ 0 & 0 & 0 & 0 & 3 \end{array}\right) \qquad \left(\begin{array}{cc|c} 1 & -1 & 3 \\ 0 & 1 & 7 \\ 0 & 0 & 0 \end{array}\right)$$

Gauss-Jordan Elimination

The RREF form of a matrix is unique.

To solve a system of equations using Gauss-Jordan elimination, we find the RREF of the augmented matrix.

Example - Gauss-Jordan elimination

$$
\begin{aligned}
x &+ 3y &+ 2z &= 2 \\
2x &+ 4y &+ 3z &= 4 \\
2x &+ y &+ 2z &= 5
\end{aligned}
$$

Example - Gauss-Jordan elimination

$$\begin{aligned} x &+ 3y &+ z &= 2 \\ 3x &+ 7y &+ z &= 4 \\ 2x &+ 5y &+ z &= 3 \end{aligned}$$

Example - Gauss-Jordan elimination

$$\begin{array}{rcrcrcr} x_1 & + & 2x_2 & + & x_3 & = & 2 \\ 2x_1 & + & 5x_2 & + & 3x_3 & = & 3 \\ 3x_1 & + & 7x_2 & + & 4x_3 & = & 6 \end{array}$$

Example time permitting

$$\begin{aligned} x &+ y &+ z &= 3 \\ x &+ 2y &+ 3z &= 7 \\ 2x &+ 3y &+ 4z &= 10 \end{aligned}$$

Example time permitting

$$\begin{aligned} x &+ 2y &+ 3z &= 6 \\ x &+ y &+ z &= 3 \\ x &- y &+ 2z &= 2 \end{aligned}$$

Example time permitting

$$\begin{aligned} 3x &+ 9y &+ 7z &= 32 \\ x &+ 3y &+ 2z &= 10 \\ 2x &+ 6y &+ 7z &= 26 \end{aligned}$$

Example time permitting

$$
\begin{array}{rcrcrcr}
2x_1 & + & 2x_2 & + & x_3 & = & 0 \\
-2x_1 & + & 5x_2 & + & 2x_3 & = & 1 \\
8x_1 & + & x_2 & + & 4x_3 & = & -1
\end{array}
$$

Example time permitting

$$\begin{array}{rcrcrcr} x_1 & + & x_2 & + & 2x_3 & = & 8 \\ -x_1 & - & 2x_2 & + & 3x_3 & = & 1 \\ 3x_1 & - & 7x_2 & + & 4x_3 & = & 10 \end{array}$$