§7.8 Electrostatic Potential

§7.8.1 Electrostatic Potential Without Sources

In Section 3.9, we associated an analytic complex potential $F(z) = V(x,y) + W(x,y)i$ with two-dimensional electrostatic fields $V(x,y)$ that were the result of three-dimensional problems in which potential was identical in every plane parallel to the xy-plane. The components of the electric field intensity $\mathbf{E}$ are the real and imaginary parts of $-\overline{F'(z)}$. Curves (cylinders) in the one-parameter family $V(x,y) = C$ are equipotentials, and curves $W(x,y) = C$ are lines of force.

In domains D which are free of charge, the potential satisfies Laplace's equation

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0. \tag{7.42a}$$

What completes the characterization of $V(x,y)$ is specification of a boundary condition on the boundary $\beta(D)$ of D, usually a Dirichlet or Neumann condition (see equations 7.22a and 7.23a).

Suppose for the moment that D is a bounded domain as shown in Figure 7.65, and it is required to solve 7.42a subject to a Dirichlet condition on $\beta(D)$,

$$V(x,y) = h(x,y), \qquad (x,y) \text{ on } \beta(D), \tag{7.42b}$$

where $h(x,y)$ is a given function. According to Corollary 3 to Theorem 4.24 in Section 4.8, the solution of this problem is unique; there cannot be two different solutions. Furthermore, Theorem 4.24 indicates that $V(x,y)$ cannot have a relative maximum or minimum inside D; maximum and minimum potentials must occur on $\beta(D)$.

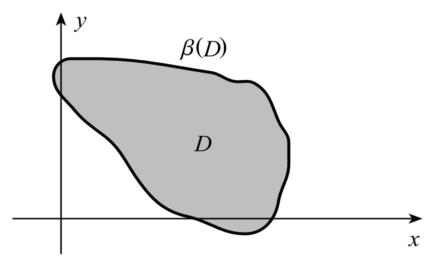


Figure 7.65

When D is a complicated domain, it may be possible to use a conformal mapping to map D onto a simpler domain, solve the simple problem, and then invert the transformation to obtain the solution to the original problem. We illustrate in the following examples.

Example 7.31 A cylindrical conductor of infinite length and radius R is centred around a line through the origin of the xy-plane and perpendicular to the plane. One half is held at potential V_1, and the other half at V_2, the parts being separated by thin pieces of insulation. Find the potential interior to the cylinder. Describe the equipotential surfaces and lines of force interior to the cylinder.

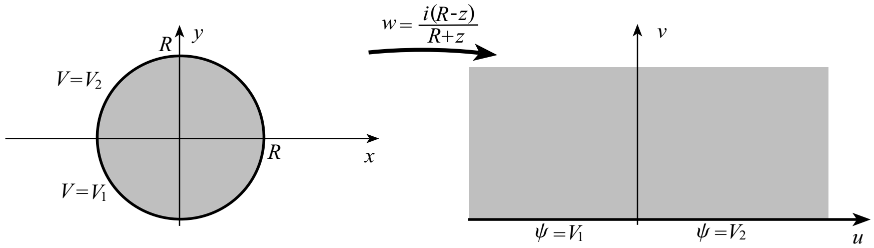


Figure 7.66

Solution Since potential in all planes parallel to the xy-plane is the same, we rephrase the problem as finding the potential inside the circle $x^2 + y^2 < R^2$, given that its values on the upper and lower halves are V_2 and V_1, respectively (Figure 7.66). We begin by mapping the circle to the half-plane $\operatorname{Im} w > 0$ with a bilinear transformation in such a way that the upper semicircle is mapped to the positive $u = \operatorname{Re} w$ axis and the lower semicircle is mapped to the negative u-axis. If we choose to map $z = R$ to $w = 0$, then $z = -R$ maps to points infinitely far out the real u-axis. Since $z = -R$ is the pole of the mapping, the transformation takes the form $w = a(z - R)/(z + R)$. If we arbitrarily demand that $z = Ri$ map to $w = 1$, then $1 = a(Ri - R)/(Ri + R)$, and this requires that $a = -i$. The bilinear transformation is therefore

$$w = \frac{i(R - z)}{R + z}.$$

According to equation 4.38, the solution $\psi(u, v)$ of Laplace's equation in the domain $\operatorname{Im} w > 0$ subject to the boundary conditions in Figure 7.66 is

$$\begin{aligned} \psi(u, v) &= \frac{V_1 + V_2}{2} + \frac{1}{\pi}(V_1 - V_2)\operatorname{Tan}^{-1}\left(-\frac{u}{v}\right) \\ &= \frac{V_1 + V_2}{2} + \frac{1}{\pi}(V_2 - V_1)\operatorname{Tan}^{-1}\left(\frac{u}{v}\right). \end{aligned}$$

If we set $w = u + vi$ and $z = x + yi$ in the bilinear transformation,

$$u + vi = \frac{i(R - x - yi)}{R + x + yi} = \frac{i[(R - x) - yi]}{(R + x) + yi}\,\frac{(R + x) - yi}{(R + x) - yi} = \frac{2yR + (R^2 - x^2 - y^2)i}{(R + x)^2 + y^2}.$$

Thus,

$$u = \frac{2yR}{(R + x)^2 + y^2}, \qquad v = \frac{R^2 - x^2 - y^2}{(R + x)^2 + y^2},$$

and the electrostatic potential is

$$V = \frac{V_1 + V_2}{2} + \frac{1}{\pi}(V_2 - V_1)\operatorname{Tan}^{-1}\left(\frac{2Ry}{R^2 - x^2 - y^2}\right).$$

Equipotential surfaces are defined implicitly by

$$C = \frac{V_1 + V_2}{2} + \frac{V_2 - V_1}{\pi}\operatorname{Tan}^{-1}\left(\frac{2Ry}{R^2 - x^2 - y^2}\right) \qquad \Longrightarrow \qquad \operatorname{Tan}^{-1}\left(\frac{2Ry}{R^2 - x^2 - y^2}\right) = k,$$

where C is a constant and $k = \pi(2C - V_1 - V_2)/(2V_2 - 2V_1)$. When we take tangents on both sides of the latter equation,

$$\frac{2Ry}{R^2 - x^2 - y^2} = \tan k \qquad \Longrightarrow \qquad R^2 - x^2 - y^2 = 2Ry \cot k.$$

This can be rearranged into the form

$$x^2 + (y + R \cot k)^2 = R^2 + R^2 \cot^2 k = R^2 \csc^2 k.$$

These are circular arcs through $(\pm R, 0)$ with centres on the y-axis ((Figure 7.67). If we set $K = \cot k$, then equipotentials are given by $x^2 + y^2 + 2KRy = R^2$. Lines of force are orthogonal trajectories of these curves. They can be derived by finding harmonic conjugates $W(x, y)$ of $V(x, y)$, but this turns out to be a formidable task. Instead, we use differential equations to find the orthogonal trajectories of $x^2 + y^2 + 2KRy = R^2$. If we differentiate this equation with respect to x, we obtain $2x + 2y\dfrac{dy}{dx} + 2KR\dfrac{dy}{dx} = 0$. Thus,

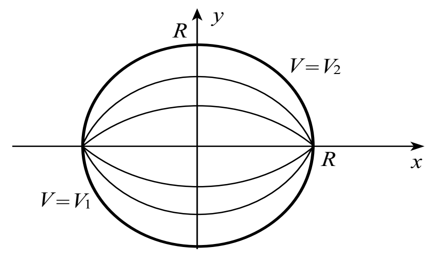


Figure 7.67

$$\frac{dy}{dx} = \frac{-x}{y + KR} = \frac{-x}{y + (R^2 - x^2 - y^2)/(2y)} = \frac{-2xy}{R^2 - x^2 + y^2}.$$

The differential equation for orthogonal trajectories is

$$\frac{dy}{dx} = \frac{R^2 - x^2 + y^2}{2xy} \qquad \Longrightarrow \qquad \frac{dy}{dx} - \frac{y}{2x} = \frac{R^2 - x^2}{2xy}.$$

We substitute $z = y^2$ and $dz/dx = 2y\, dy/dx$ into this Bernoulli equation,

$$\frac{1}{2y}\frac{dz}{dx} - \frac{y}{2x} = \frac{R^2 - x^2}{2xy} \qquad \Longrightarrow \qquad \frac{dz}{dx} - \frac{z}{x} = \frac{R^2 - x^2}{x}.$$

An integrating factor for this linear first-order differential equation is $e^{\int (-1/x)\, dx} = 1/x$. When the differential equation is multiplied by $1/x$,

$$\frac{1}{x}\frac{dz}{dx} - \frac{z}{x^2} = \frac{R^2 - x^2}{x^2} \qquad \Longrightarrow \qquad \frac{d}{dx}\left(\frac{z}{x}\right) = \frac{R^2 - x^2}{x^2}.$$

Integration gives

$$\frac{z}{x} = -\frac{R^2}{x} - x + 2C \qquad \Longrightarrow \qquad y^2 = -R^2 - x^2 + 2Cx,$$

where C is a constant. Lines of force are therefore $(x - C)^2 + y^2 = C^2 - R^2$. These are circular arcs with centres on the x-axis.●

Example 7.32 Find potential in the semi-infinite strip $-1 < x < 1$, $y > 0$ if potential on the horizontal side is 0 and that on the vertical sides is $V_0 > 0$. Identify and plot equipotentials.

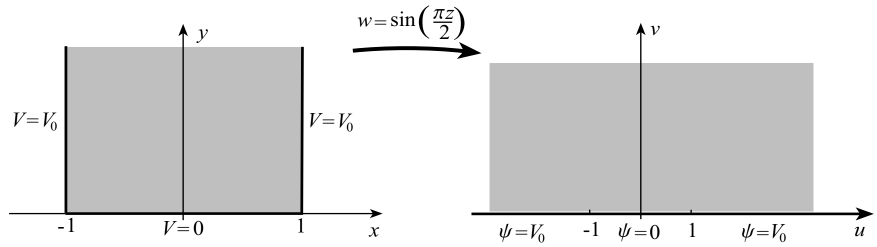


Figure 7.68

Solution In Examples 7.13 and 7.22 we derived the mapping $w = \sin[\pi z/(2)]$ that maps the strip to the half-plane $\operatorname{Im} w > 0$ in such a way that $z = \pm 1$ are mapped to $w = \pm 1$. Formula 4.38 gives the solution $\psi(u, v)$ of Laplace's equation in $\operatorname{Im} w > 0$ subject to the piecewise constant boundary condition along $\operatorname{Im} w = 0$,

$$\begin{aligned}\psi(u,v) &= \frac{1}{2}(V_0 + V_0) + \frac{1}{\pi}\left[V_0 \operatorname{Tan}^{-1}\left(\frac{-1-u}{v}\right) - V_0 \operatorname{Tan}^{-1}\left(\frac{1-u}{v}\right)\right]\\ &= V_0 - \frac{V_0}{\pi}\left[\operatorname{Tan}^{-1}\left(\frac{1+u}{v}\right) + \operatorname{Tan}^{-1}\left(\frac{1-u}{v}\right)\right].\end{aligned}$$

Since $w = u + vi = \sin\frac{\pi z}{2} = \sin\frac{\pi x}{2}\cosh\frac{\pi y}{2} + \cos\frac{\pi x}{2}\sinh\frac{\pi y}{2}\, i$, it follows that

$$\begin{aligned}V(x,y) &= \psi[u(x,y), v(x,y)]\\ &= V_0 - \frac{V_0}{\pi}\left[\operatorname{Tan}^{-1}\left(\frac{1+\sin\frac{\pi x}{2}\cosh\frac{\pi y}{2}}{\cos\frac{\pi x}{2}\sinh\frac{\pi y}{2}}\right) + \operatorname{Tan}^{-1}\left(\frac{1-\sin\frac{\pi x}{2}\cosh\frac{\pi y}{2}}{\cos\frac{\pi x}{2}\sinh\frac{\pi y}{2}}\right)\right].\end{aligned}$$

We can simplify this solution by bringing the inverse tangents together. If we take tangents of both sides of

$$\frac{\pi}{V_0}(V_0 - \psi) = \operatorname{Tan}^{-1}\left(\frac{1+u}{v}\right) + \operatorname{Tan}^{-1}\left(\frac{1-u}{v}\right),$$

we obtain

$$\tan\left[\frac{\pi}{V_0}(V_0 - \psi)\right] = \frac{\frac{1+u}{v} + \frac{1-u}{v}}{1 - \left(\frac{1+u}{v}\right)\left(\frac{1-u}{v}\right)} = \frac{2v}{u^2 + v^2 - 1}.$$

Since $0 < \psi < V_0$, it follows that $0 < \pi(V_0 - \psi)/V_0 < \pi$. With $\tan^{-1}$ denoting values of the inverse tangent function between 0 and π, the solution can be written in the form

$$\frac{\pi}{V_0}(V_0 - \psi) = \tan^{-1}\left(\frac{2v}{u^2+v^2-1}\right) \quad \Longrightarrow \quad \psi(u,v) = V_0 - \frac{V_0}{\pi}\tan^{-1}\left(\frac{2v}{u^2+v^2-1}\right).$$

Since $u = \sin\frac{\pi x}{2}\cosh\frac{\pi y}{2}$ and $v = \cos\frac{\pi x}{2}\sinh\frac{\pi y}{2}$,

$$V(x,y) = V_0 - \frac{V_0}{\pi}\tan^{-1}\left[\frac{2\cos\frac{\pi x}{2}\sinh\frac{\pi y}{2}}{\sin^2\frac{\pi x}{2}\cosh^2\frac{\pi y}{2} + \cos^2\frac{\pi x}{2}\sinh^2\frac{\pi y}{2} - 1}\right]$$

$$= V_0 - \frac{V_0}{\pi}\tan^{-1}\left[\frac{2\cos\dfrac{\pi x}{2}\sinh\dfrac{\pi y}{2}}{\sin^2\dfrac{\pi x}{2}+\sinh^2\dfrac{\pi y}{2}-1}\right]$$

$$= V_0 - \frac{V_0}{\pi}\tan^{-1}\left[\frac{2\cos\dfrac{\pi x}{2}\sinh\dfrac{\pi y}{2}}{\sinh^2\dfrac{\pi y}{2}-\cos^2\dfrac{\pi x}{2}}\right].$$

Equipotentials are defined implicitly by

$$C = V_0 - \frac{V_0}{\pi}\tan^{-1}\left[\frac{2\cos\dfrac{\pi x}{2}\sinh\dfrac{\pi y}{2}}{\sinh^2\dfrac{\pi y}{2}-\cos^2\dfrac{\pi x}{2}}\right], \quad \text{or,} \quad \tan\left[\frac{\pi(V_0-C)}{V_0}\right] = \frac{2\cos\dfrac{\pi x}{2}\sinh\dfrac{\pi y}{2}}{\sinh^2\dfrac{\pi y}{2}-\cos^2\dfrac{\pi x}{2}}.$$

They are shown in Figure 7.69.●

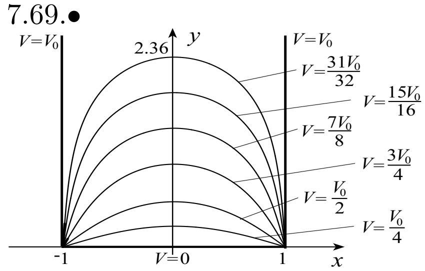


Figure 7.69

Example 7.33 An infinite conducting cylinder with radius 2 has its centre 4 units away from a plane that is parallel to the axis of the cylinder. The cylinder is held at potential V_1 while the plane is at potential V_0. Find potential at all points outside the cylinder on the same side of the plane as the cylinder.

Solution Figure 7.70 shows a cross-section of the cylinder which therefore has its axis perpendicular to the xy-plane. The plane at potential V_0 is represented by the x-axis. To find the potential above the x-axis and outside the circle, we use the technique of Example 7.12 to map the region to an annulus. Consider finding a pair of points z_1 and z_2 that are simultaneously inverses with respect to the x-axis and to the circle . The inverse of z_1 with respect to the x-axis is its conjugate $\overline{z_1}$. According to equation 7.11, z_1 and $\overline{z_1}$ are inverses with respect to the circle if

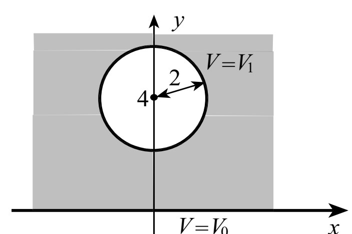


Figure 7.70

$$\overline{z_1} = 4i + \frac{4}{\overline{z_1}-(-4i)}.$$

Solutions of this equation are $z_1 = \pm 2\sqrt{3}i$. The bilinear transformation $w = f(z) = \dfrac{z-2\sqrt{3}i}{z+2\sqrt{3}i}$ maps the region outside the cylinder to an annulus (Figure 7.71). The radius of the image of the circle $|z-4i| = 2$ is the modulus of the image of any point on the circle; in particular,

$$|f(2i)| = \left|\frac{2i - 2\sqrt{3}i}{2i + 2\sqrt{3}i}\right| = 2 - \sqrt{3}.$$

The radius of the image of the x-axis is the modulus of the image of any point on the axis; in particular,

$$|f(0)| = \left|\frac{-2\sqrt{3}i}{2\sqrt{3}i}\right| = 1.$$

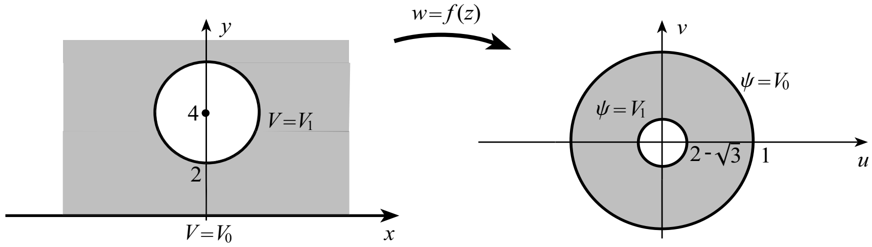


Figure 7.71

Any function of the form $c+d\ln(u^2+v^2)$ satisfies Laplace's equation, and this function is constant along any circle centred at the origin. If $\psi(u,v) = c + d\ln(u^2+v^2)$ is to be the solution of Laplace's equation in the annulus satisfying the specified boundary conditions, then c and d are given by the equations

$$V_1 = c + d\ln(2-\sqrt{3})^2, \quad V_0 = c \qquad \Longrightarrow \qquad d = \frac{V_1 - V_0}{2\ln(2-\sqrt{3})}.$$

Thus,

$$\psi(u,v) = V_0 + \frac{V_1 - V_0}{2\ln(2-\sqrt{3})}\ln(u^2+v^2).$$

To express this in terms of x and y, we could find u and v in terms of x and y. A simpler expression is obtained if we note that $u^2+v^2 = |w|^2$, and therefore

$$\begin{aligned} V(x,y) = V_0 + \frac{V_1 - V_0}{2\ln(2-\sqrt{3})}\ln|w|^2 &= V_0 + \frac{V_1 - V_0}{2\ln(2-\sqrt{3})}\ln\left|\frac{z - 2\sqrt{3}i}{z + 2\sqrt{3}i}\right|^2 \\ &= V_0 + \frac{V_1 - V_0}{2\ln(2-\sqrt{3})}\ln\left[\frac{x^2 + (y-2\sqrt{3})^2}{x^2 + (y+2\sqrt{3})^2}\right]. \bullet \end{aligned}$$

Example 7.34 In Figure 7.72a, the horizontal lines $y = \pm d$ for $x < 0$ represent the cross-section of a semi-infinite parallel plate capacitor. If potentials on top and bottom plates are V_0 and $-V_0$, respectively, find equipotentials and lines of force.

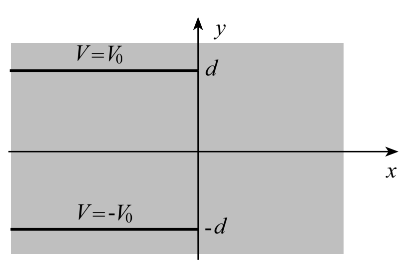


Figure 7.72a

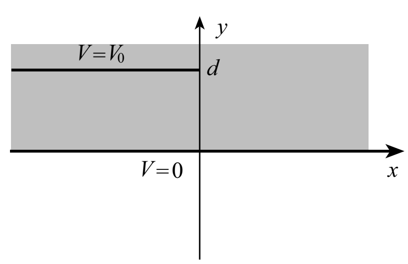


Figure 7.72b

Solution Symmetry indicates that potential along the x-axis is zero for $x < 0$ and $x > 0$, and that potential below the x-axis is the negative of that above the x-axis. We therefore solve the problem in Figure 7.72b where potential is V_0 along the half-line $y = d$ $(x < 0)$ and is 0 along $y = 0$ $(-\infty < x < \infty)$. We begin by mapping the region onto the upper half-plane $\operatorname{Im}\tilde{w} > 0$ with a Schwarz-Christoffel transformation. We regard the region in the z-plane as a degenerate triangle with vertices z_1, z_2, and z_3 in Figure 7.73.

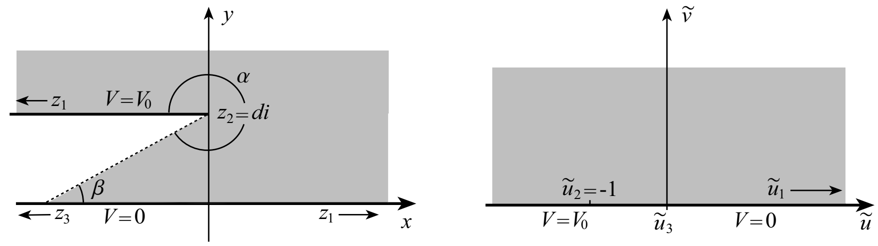


Figure 7.73

Angles α and β at z_2 and z_3 must be made to approach 2π and 0 respectively. Difficulty with the angle at z_1 is eliminated by choosing $\tilde{u}_1$, the pre-image of z_1 as the point at infinity, in which case the angle does not enter the Schwarz-Christoffel transformation. If we choose $\tilde{u}_2 = -1$ and $\tilde{u}_3 = 0$, then taking limits of

$$\frac{dz}{d\tilde{w}} = A(\tilde{w}+1)^{\alpha/\pi-1}\tilde{w}^{\beta/\pi-1}$$

as $\alpha \to 2\pi$ and $\beta \to 0$ gives

$$\frac{dz}{d\tilde{w}} = A\left(\frac{\tilde{w}+1}{\tilde{w}}\right) = A\left(1+\frac{1}{\tilde{w}}\right).$$

Integration gives

$$z = A(\tilde{w} + \log_\phi \tilde{w}) + B.$$

If we choose $\phi = -\pi/2$, then for $\tilde{u}_2 = -1$ to map to $z_2 = di$, A and B must satisfy

$$di = A[-1 + \log_{-\pi/2}(-1)] + B = A(-1+\pi i) + B.$$

This is satisfied if we choose $A = B = d/\pi$, in which case

$$z = \frac{d}{\pi}(1 + \tilde{w} + \log_{-\pi/2}\tilde{w}).$$

It is straightforward to check that this transformation maps the positive $\tilde{u}$-axis to the x-axis, and maps both parts $-\infty < \tilde{u} < -1$ and $-1 < \tilde{u} < 0$ of the negative $\tilde{u}$-axis to the half-line $y = d$, $x < 0$.

We could now use equation 4.38 to find potential in the half-plane $\operatorname{Im}\tilde{w} > 0$ with value V_0 on the negative $\tilde{u}$-axis and value 0 on the positive $\tilde{u}$-axis. Unfortunately, it would not yield a convenient representation for equipotentials and/or lines of force. Instead, we map $\operatorname{Im}\tilde{w} > 0$ to the infinite strip in Figure 7.74 in such a way that the negative $\tilde{u}$-axis is mapped to the line $V = V_0 i$ and the positive $\tilde{u}$-axis is mapped to the u-axis. A mapping that will do this is $w = (V_0/\pi)\log_{-\pi/2}\tilde{w}$.

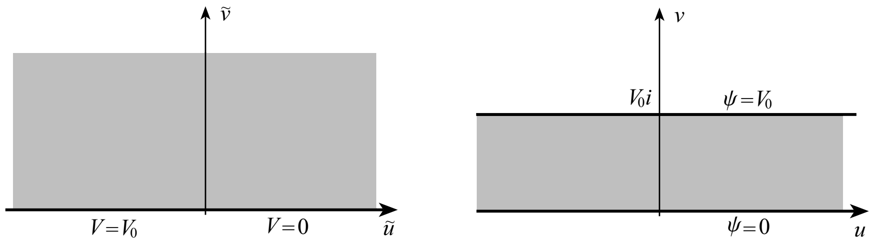


Figure 7.74

Since $\tilde{w} = e^{\pi w/V_0}$, the transformation

$$z = \frac{d}{\pi}\left[1 + e^{\pi w/V_0} + \log_{-\pi/2}(e^{\pi w/V_0})\right] = \frac{d}{\pi}\left(1 + e^{\pi w/V_0} + \frac{\pi w}{V_0}\right)$$

therefore maps the strip in Figure 7.74 to the half-space, with u-axis mapped to x-axis, and line $V_0 i$ mapped to half-line $y = d$, $x < 0$.

The solution of Laplace's equation in the strip is $\psi(u, v) = v$. To find potential $V(x, y)$, we would set

$$x + yi = z = \frac{d}{\pi}\left[1 + e^{\pi(u+vi)/V_0} + \frac{\pi(u + vi)}{V_0}\right],$$

and solve for v in terms of x and y, an impossibility. We can, however, find equations for equipotential curves and lines of force. If we take take real and imaginary parts of the above equation,

$$x = \frac{d}{\pi}\left[1 + e^{\pi u/V_0}\cos\frac{\pi v}{V_0} + \frac{\pi u}{V_0}\right], \qquad \text{and} \qquad y = \frac{d}{\pi}\left[e^{\pi u/V_0}\sin\frac{\pi v}{V_0} + \frac{\pi v}{V_0}\right].$$

Equipotential curves for $y \geq 0$ are therefore defined parametrically by

$$x = \frac{d}{\pi}\left[1 + e^{\pi u/V_0}\cos\frac{\pi V}{V_0} + \frac{\pi u}{V_0}\right], \qquad y = \frac{d}{\pi}\left[e^{\pi u/V_0}\sin\frac{\pi V}{V_0} + \frac{\pi V}{V_0}\right],$$

for fixed $V \geq 0$ and u as parameter.

Orthogonal trajectories in the strip are $u = U =$ constant, and therefore lines of force in the xy-plane are given parametrically by

$$x = \frac{d}{\pi}\left[1 + e^{\pi U/V_0}\cos\frac{\pi v}{V_0} + \frac{\pi U}{V_0}\right], \qquad y = \frac{d}{\pi}\left[e^{\pi U/V_0}\sin\frac{\pi v}{V_0} + \frac{\pi v}{V_0}\right],$$

for fixed U and v as parameter. Both sets of curves are shown in Figure 7.75.●

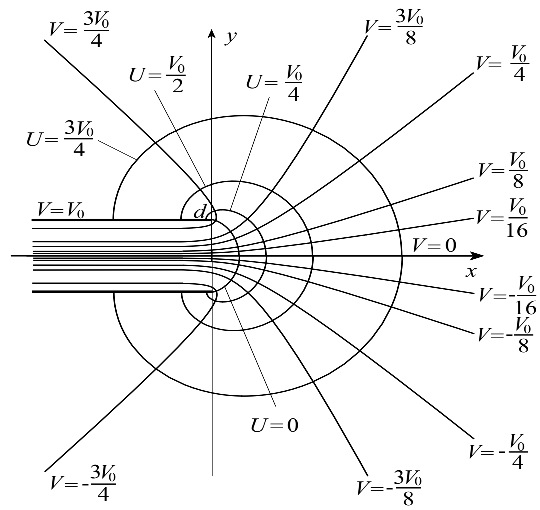


Figure 7.75

EXERCISES 7.8.1

For simplicity in formulation, exercises will be posed in the xy-plane. They are, however, the result of three-dimensional problems for which potential is the same in every plane parallel to the xy-plane.

1. Find potential interior to the wedge $0 \leq \theta \leq \alpha < 2\pi$ if potential along $\theta = 0$, $r > 0$ is V_0, and along $\theta = \alpha$, $r > 0$, potential is V_α.

2. Find potential in the semicircle $x^2 + y^2 < R^2$, $y > 0$ when $V = V_1$ on $y = 0$ and $V = V_0$ on $x^2 + y^2 = R^2$. Hint: Use the mapping $w = i(R - z)/(R + z)$.

3. Find potential in the circle $x^2 + y^2 < R^2$ when potential on that part of the circle in the first quadrant is V_0 and potential on the remainder of the circle is zero. Hint: Use a bilinear transformation that maps the points $z = Ri$, $z = -R$, and $z = R$ to $w = 0$, $w = 1$, and $w = \infty$. What is the potential at the centre of the circle?

4. Find potential in the half-plane $y > 0$ outside the circle $x^2 + y^2 = R^2$ given that potential on the semicircle is V_0 and potential on the x-axis for $|x| > R$ is zero. Hint: Use the bilinear transformation $w = (z - R)/(R + z)$ and Exercise 1.

5. Find potential in the half-plane $y > 0$ outside the circle $x^2 + y^2 = R^2$ given that potential on the semicircle is V_1 and potential on the x-axis for $|x| > R$ is V_0. Hint: Use the bilinear transformation $w = (z - R)/(R + z)$ and Exercise 1.

6. Find potential in the semi-infinite strip $-a < x < a$, $y > 0$ if potential on the horizontal side is V_0 and that on the vertical sides is V_1.

7. Find potential in the semi-infinite strip $-a < y < a$, $x > 0$ if potential on the horizontal sides is V_0 and that on the vertical side is V_1. Hint: Use the transformation $w = i \sinh[\pi z/(2a)]$.

8. Find potential in the domain bounded by that part of $x^2 + y^2 = a^2$ above the x-axis and that part of $x^2 + (y - a)^2 = 2a^2$ below the x-axis. Potential on $x^2 + y^2 = a^2$ is V_0, and potential on

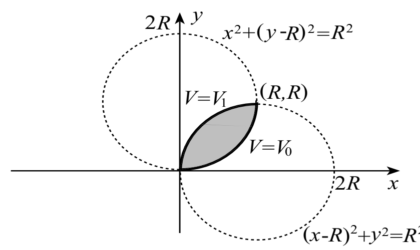
$x^2 + (y - a)^2 = 2a^2$ is V_1. Hint: First map the domain to the sector $\pi/2 < \arg w^* < 5\pi/4$ by $w^* = (z - a)/(z + a)$, and then map the sector to an infinite strip with $w = \log_{-\pi/2} w^* - \pi i/2$.

9. Find potential at points in the half-plane $y > 0$ that are outside the circle $x^2 + (y - a)^2 = a^2$ given that potential on the circle is a constant V_0 and potential on $y = 0$ is a constant V_1. Hint: Try $w = -2a/z$. Find equipotential curves.

10. A circle with radius R has its centre $a > R$ units away from a line. The circle is held at potential V_1 while the line is at potential V_0. Find potential at all points outside the circle on the same side of the line as the circle. Hint: Use the bilinear transformation in Exercise 28 of Section 7.2 to map the problem to that in an annulus.

11. Two conducting circles have equations $x^2 + (y - 2)^2 = 4$ and $x^2 + y^2 = 25$. The inner circle is held at potential V_0, and the outer one is held at potential V_1. Find the potential between the circles. Hint: Use a bilinear transformation like that in Exercise 29 of Section 7.2 to map the region to an annulus.

12. Two arcs are parts of the circles $x^2 + (y - R)^2 = R^2$ and $(x - R)^2 + y^2 = R^2$ (figure to the right). The lower arc is held at potential V_0, and the upper one is held at potential V_1. Find the potential between the arcs. Hint: First, use a bilinear transformation with $z = R(1 + i)$ as pole and $z = 0$ as zero. Then map the image to the first quadrant and use Exercise 19 of Section 7.5.

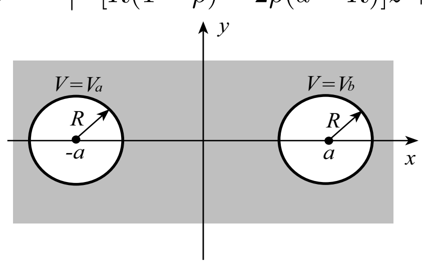


13. Two circles $(x - a)^2 + y^2 = a^2$ and $(x - b)^2 + y^2 = b^2$, where $b > a$, are held at constant potentials V_0 and $V_1 > V_0$, respectively. The point of contact of the circles is separated by perfect insullation.
(a) Find potential between the circles.
(b) Find and draw equipotential curves.

14. The x-axis to the left of $x = -1$ is held at constant potential $V = V_0$, and the x-axis to the right of $x = 1$ is held at potential $V = 0$. Find the resulting potential in the xy-plane. Hint: Consider the mapping $w = \mathrm{Cos}^{-1} z$, and see Exercises 43 and 38 in Section 3.7. Find and draw equipotential curves.

15. Use the transformation of Exercise 33 in Section 7.2 to show that the potential in the region exterior to the two circles in the figure below can be expressed in the form

$$V(x, y) = V_a + \frac{V_b - V_a}{\ln \rho} \ln \left| \frac{[R\rho(1 - \rho) + 2\rho(a - R)]z + [R\rho(1 - \rho)(a - R) - 2\rho(a^2 - R^2)]}{[R(1 - \rho) - 2\rho(a - R)]z + R(1 - \rho)(a - R) + 2\rho(a^2 - R^2)]} \right|.$$



§7.8.2 Electrostatic Potential With Sources

In the previous subsection, electrostatic potential in a domain D was the result of surfaces bounding D being held at constant potentials. In this subsection we introduce line charges as additional factors affecting potential. To begin with, consider a line of charge q coulombs per metre perpendicular to the xy-plane at the origin (Figure 7.76). We use Coulomb's law to find the potential in space due to the line of charge rather than working with Laplace's equation.

The electric field intensity $\mathbf{E}$ due to this line of charge is always parallel to the xy-plane, and is therefore a function of x and y only. By definition, $\mathbf{E}$ is the force on a unit positive charge placed in the field. With Coulomb's law, we obtain for the magnitude of $\mathbf{E}$ at (x,y),

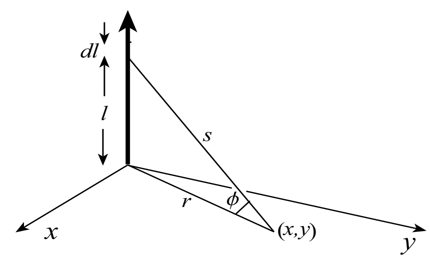


Figure 7.76

$$|\mathbf{E}| = 2\int_0^\infty \cos\phi\,\frac{q(1)}{4\pi\epsilon_0 s^2}dl = \frac{rq}{2\pi\epsilon_0}\int_0^\infty \frac{1}{(r^2+l^2)^{3/2}}dl.$$

Since $l = r\,\tan\phi$, we change variables of integration,

$$|\mathbf{E}| = \frac{rq}{2\pi\epsilon_0}\int_0^{\pi/2}\frac{r\sec^2\phi}{r^3\sec^3\phi}d\phi = \frac{q}{2\pi\epsilon_0 r}\int_0^{\pi/2}\cos\phi\,d\phi = \frac{q}{2\pi\epsilon_0 r} = \frac{q}{2\pi\epsilon_0\sqrt{x^2+y^2}}.$$

Since $\mathbf{E}$ is directed radially away from the line of charge,

$$\mathbf{E} = \frac{q}{2\pi\epsilon_0\sqrt{x^2+y^2}}\,\frac{x\hat{\mathbf{i}}+y\hat{\mathbf{j}}}{\sqrt{x^2+y^2}} = \frac{q}{2\pi\epsilon_0(x^2+y^2)}(x\hat{\mathbf{i}}+y\hat{\mathbf{j}}).$$

Because the electric field is the negative of the gradient of the potential ($\mathbf{E} = -\nabla V$, see equation 3.61), it follows that

$$\frac{\partial V}{\partial x} = -\frac{qx}{2\pi\epsilon_0(x^2+y^2)}, \qquad \frac{\partial V}{\partial y} = -\frac{qy}{2\pi\epsilon_0(x^2+y^2)}.$$

These give

$$V(x,y) = \frac{-q}{4\pi\epsilon_0}\ln(x^2+y^2) + D, \tag{7.43}$$

where D is a real constant, as the real potential due to a line of charge q coulombs per metre at the origin. We know that the function $\ln\sqrt{x^2+y^2}$ is the real part of the complex function $\log_\phi z$. Hence, the complex potential function associated with a line charge q coulombs per metre at the origin is

$$F(z) = \frac{-q}{2\pi\epsilon_0}\log_\phi z + (D+Ei). \tag{7.44}$$

When the line of charge is at the point (x_0,y_0), the real potential is

$$V(x,y) = \frac{-q}{4\pi\epsilon_0}\ln[(x-x_0)^2+(y-y_0)^2] + D, \tag{7.45}$$

and the complex electrostatic potential is

$$F(z) = \frac{-q}{2\pi\epsilon_0} \log_\phi (z - z_0) + (D + Ei), \tag{7.46}$$

where $z_0 = x_0 + y_0 i$. In the following examples, we introduce multiple line charges into space and/or the presence of bounding surfaces.

Example 7.35 A line charge has q coulombs per metre. If potential at a distance R from the charge is V_R, find potential at a distance r from the charge.

Solution According to equation 7.45, potential at a point (x, y) due to a line charge at (x_0, y_0) is

$$V(x, y) = \frac{-q}{4\pi\epsilon_0} \ln [(x - x_0)^2 + (y - y_0)^2] + D.$$

Since potential on the circle $(x - x_0)^2 + (y - y_0)^2 = R^2$ is V_R,

$$V_R = \frac{-q}{4\pi\epsilon_0} \ln (R^2) + D \qquad \Longrightarrow \qquad D = V_R + \frac{k}{2\pi\epsilon_0} \ln R.$$

Thus,

$$V(x, y) = \frac{-q}{4\pi\epsilon_0} \ln \left[\frac{(x - x_0)^2 + (y - y_0)^2}{R^2} \right] + V_R.$$

If $V(r)$ represents potential at a distance r from the source, then

$$V(r) = V_R - \frac{q}{2\pi\epsilon_0} \ln (r/R). \bullet$$

It is important to realize that the development of equations 7.43–7.46 and the discussion in Example 7.35 took place in all space. There were no boundaries to consider that might affect potential; potential was due only to a line of charge. In the presence of boundaries at designated potentials, equations 7.43–7.46 might, or might not, be a correct description of potential. For instance, suppose that in Example 7.35, a circular cylinder of radius R at potential V_R has the line charge along its axis. Because the cylinder is coincident with an equipotential surface for the line charge, potential inside the cylinder is exactly as in the example. In other words, the presence of this particular bounding surface does not affect the potential function, only its domain of definition. On the other hand, if the line charge is outside the cylinder, finding potential outside the cylinder is more difficult (see Exercise 3).

In preparation for the next example, we note that the real (or complex) potential function due to multiple line charges (in space) is simply the sum of their real (or, complex) potential functions.

Example 7.36 Find the real electrostatic potential due to a line of charge q coulombs per metre perpendicular to the xy-plane at (x_0, y_0), and a line of charge $-q$ coulombs per metre perpendicular to the xy-plane at (x_1, y_1). Find equipotentials and lines of force when $y_0 = y_1 = 0$ and $x_0 = -x_1 = a > 0$.

Solution If we add potentials due to the line charges, and set $z_0 = x_0 + y_0 i$ and $z_1 = x_1 + y_1 i$, equation 7.46 gives the complex electrostatic potential as

$$F(z) = \frac{q}{2\pi\epsilon_0}[\log_\psi(z - z_1) - \log_\phi(z - z_0)] + (D + Ei).$$

The real electrostatic potential is

$$\begin{aligned} V(x,y) = \text{Re}[F(z)] &= \frac{q}{2\pi\epsilon_0}[\ln\sqrt{(x-x_1)^2+(y-y_1)^2} - \ln\sqrt{(x-x_0)^2+(y-y_0)^2}] + D \\ &= \frac{q}{4\pi\epsilon_0}\ln\left[\frac{(x-x_1)^2+(y-y_1)^2}{(x-x_0)^2+(y-y_0)^2}\right] + D. \end{aligned}$$

Equipotentials are defined by

$$\frac{q}{4\pi\epsilon_0}\ln\left[\frac{(x-x_1)^2+(y-y_1)^2}{(x-x_0)^2+(y-y_0)^2}\right] = C = \text{constant},$$

and this equation can be expressed in the form

$$\frac{(x-x_1)^2+(y-y_1)^2}{(x-x_0)^2+(y-y_0)^2} = A, \qquad A = e^{4\pi\epsilon_0 C/q}.$$

For $y_0 = y_1 = 0$ and $x_0 = -x_1 = a$, this equation reduces to

$$\frac{(x+a)^2+y^2}{(x-a)^2+y^2} = A.$$

When $A = 1$, the equipotential becomes the line $x = 0$. When $A > 1$, the equipotential can be written in the form

$$\left[x - a\left(\frac{A+1}{A-1}\right)\right]^2 + y^2 = \left[\left(\frac{A+1}{A-1}\right)^2 - 1\right]a^2,$$

a circle with centre on the positive x-axis enclosing charge q at $(a, 0)$. When $A < 1$, the equipotential reduces to

$$\left[x + a\left(\frac{1+A}{1-A}\right)\right]^2 + y^2 = \left[\left(\frac{1+A}{1-A}\right)^2 - 1\right]a^2,$$

a circle with centre on the negative x-axis enclosing charge $-q$ at $(-a, 0)$. We have shown some of these equipotentials in Figure 7.77.

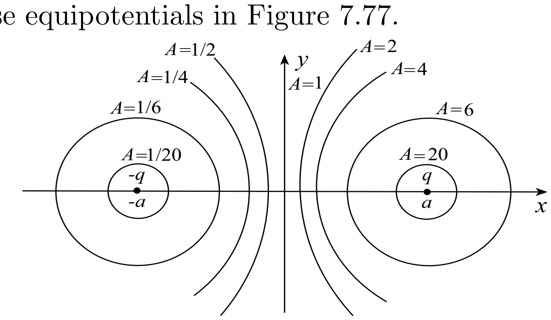


Figure 7.77

Lines of force are the orthogonal trajectories of these two families of circles. They are defined by $\text{Im}\,[F(z)] = \text{constant}$; that is

$$\frac{q}{2\pi\epsilon_0}[\arg_\psi(z - z_1) - \arg_\phi(z - z_0)] = \text{constant}.$$

This equation can be expressed in the form

$$\tan^{-1}\left(\frac{y-y_1}{x-x_1}\right) - \tan^{-1}\left(\frac{y-y_0}{x-x_0}\right) = \text{constant} = B.$$

For $y_0 = y_1 = 0$ and $x_0 = -x_1 = a$,

$$\tan^{-1}\left(\frac{y}{x+a}\right) - \tan^{-1}\left(\frac{y}{x-a}\right) = B.$$

Taking tangents of both sides of this equation gives

$$\frac{\dfrac{y}{x+a} - \dfrac{y}{x-a}}{1 + \dfrac{y^2}{x^2-a^2}} = \tan B,$$

and this simplifies to $D(x^2+y^2-a^2) = -2ay$, where $D = \tan B$. When $D = 0$, we obtain $y = 0$. When $D \neq 0$, we rewrite the equation in the form

$$x^2 + \left(y + \frac{a}{D}\right)^2 = \frac{a^2}{D^2}(D^2+1).$$

These are circles passing through q and $-q$ (Figure 7.78).●

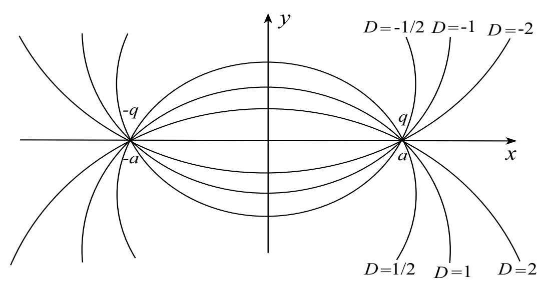


Figure 7.78

The following example leads to a technique that is often useful in handling line charges in the presence of conducting boundaries.

Example 7.37 A line charge of q coulombs per metre is located at the point (x_0, y_0) and is perpendicular to the xy-plane. A plane containing the x-axis, and perpendicular to the xy-plane, is held at potential V_0. Find potential in the half space $y > 0$. Find and draw equipotentials.

Solution We model the three-dimensional situation with the x-axis representing the plane and a point source at (x_0, y_0) (Figure 7.79). To make use of complex potential function 7.44 for potential due to a source in the absence of boundaries, we map the half-plane $y > 0$ to the interior of the unit circle $|w| = 1$ in such a way that the source is mapped to the origin and the x-axis is mapped to the unit circle. Although the circle $|w| = 1$ is a boundary for the region, the fact that it is at constant potential makes it compatible with a source at the origin with no boundary where equipotentials are circles centred at the origin. A bilinear transformation will perform the mapping. According to Exercise 24 in Section 7.2, all bilinear transformations that map the half-plane $y > 0$ to the unit circle and take a point z_0 to $w = 0$ are of the form $w = e^{\lambda i}(z - z_0)/(z - \overline{z_0})$, for some real constant λ. If we arbitrarily set $\lambda = 0$ and choose $z_0 = x_0 + y_0 i$, then

$$w = \frac{z - z_0}{z - \overline{z_0}}.$$

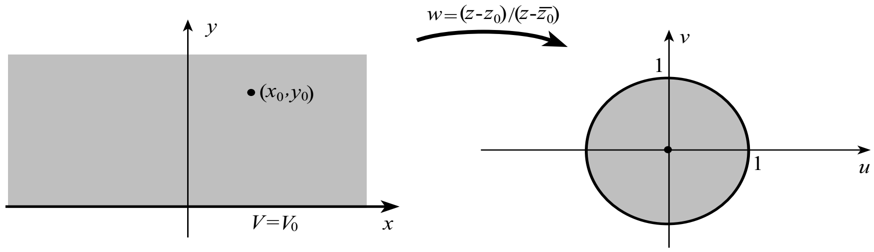


Figure 7.79

The complex potential function in the w-plane due to the source at the origin must be of the form

$$G(w) = -\frac{q}{2\pi\epsilon_0}\log_\phi w + (D + Ei),$$

and therefore the complex potential function in the z-plane is

$$F(z) = -\frac{q}{2\pi\epsilon_0}\log_\phi\left(\frac{z - z_0}{z - \overline{z_0}}\right) + (D + Ei).$$

The real potential function is

$$V(x,y) = \mathrm{Re}\,[F(z)] = \frac{q}{2\pi\epsilon_0}\ln\left|\frac{z - \overline{z_0}}{z - z_0}\right| + D.$$

Since potential along $y = 0$ is V_0, we must have

$$V_0 = \frac{q}{2\pi\epsilon_0}\ln\left|\frac{x - x_0 + y_0 i}{x - x_0 - y_0 i}\right| + D = D.$$

Thus,

$$V(x,y) = V_0 + \frac{q}{4\pi\epsilon_0}\ln\left[\frac{(x - x_0)^2 + (y + y_0)^2}{(x - x_0)^2 + (y - y_0)^2}\right].$$

Equipotential curves are defined implicitly by

$$C = V_0 + \frac{q}{4\pi\epsilon_0}\ln\left[\frac{(x - x_0)^2 + (y + y_0)^2}{(x - x_0)^2 + (y - y_0)^2}\right].$$

If we define $k = e^{2\pi\epsilon_0(C - V_0)/q}$, it is a straightforward exercise in algebra to show that this equation can be rewritten in the form

$$(x - x_0)^2 + (y - y_0\coth k)^2 = y_0^2\mathrm{csch}^2 k.$$

These are circles with centres at $(x_0, y_0\coth k)$ some of which are shown in Figure 7.80.•

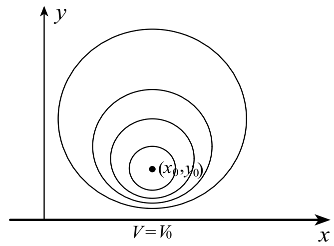


Figure 7.80

It is instructive to write the complex potential function in this example in the following form

$$F(z) = V_0 - \frac{q}{2\pi\epsilon_0}\log_\phi(z - z_0) + \frac{q}{2\pi\epsilon_0}\log_\phi(z - \overline{z_0}) + Ei.$$

It is V_0 plus the sum of two complex potential functions, one due to a positive line charge at z_0 and the other due to a negative line charge at $\overline{z_0}$. The two charges result in a potential of zero along the x-axis. This suggests an alternative technique for solving Example 7.37, a technique called the *method of images*. The given problem with a charge and a boundary with prescribed potential is replaced by a problem with the original charge and a second charge so that the two together yield potential zero on the boundary. To this is added the constant potential V_0. In the example, the negative charge is at the image of the positive charge in the x-axis. In some configurations, more than one image charge may be required, or the image may be in some other line, or even a circle (see Exercise 2 for the case of a circle). We use a conformal mapping and the method of images in the following example.

Example 7.38 Two infinite parallel plates separated by a distance a are both at potential V_0. If a line charge of q coulombs per metre is a distance b from one of the plates, determine the electrostatic potential between the plates. Find and plot equipotentials.

Solution We reduce the problem to two dimensions by modelling the plates as the lines $y = 0$ and $y = a$. The line charge passes through the point $(0, b)$ on the y-axis (Figure 7.81).

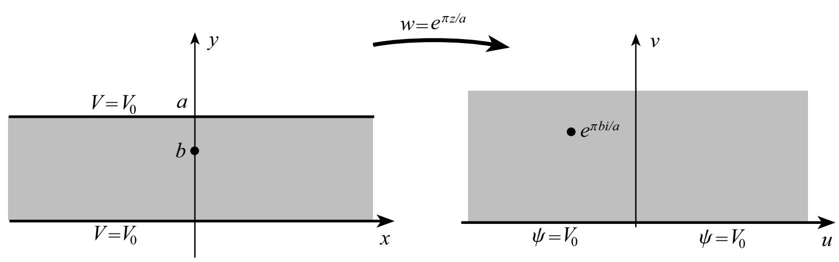


Figure 7.81

The transformation $w = e^{\pi z/a}$ maps the strip between the plates to the half-plane $\operatorname{Im} w > 0$, with the point $z = b$ mapped to $w = e^{\pi bi/a}$. To solve the potential problem in the w-plane, we mometarily replace the boundary condition $\psi = V_0$ with $\psi = 0$. A charge configuration of q at $w = e^{\pi bi/a}$ and $-q$ at the complex conjugate point $w = e^{-\pi bi/a}$ produces zero potential on the real w-axis. Consequently, the complex electrostatic potential in the upper-half of the w-plane when the boundary condition is $V = 0$ is

$$G(w) = -\frac{q}{2\pi\epsilon_0}\log_\phi\left(w - e^{\pi bi/a}\right) + \frac{q}{2\pi\epsilon_0}\log_\psi\left(w - e^{-\pi bi/a}\right) + (D + Ei).$$

The complex potential when the boundary condition is $V = V_0$ is then

$$G(w) = V_0 - \frac{q}{2\pi\epsilon_0}\log_\phi\left(w - e^{\pi bi/a}\right) + \frac{q}{2\pi\epsilon_0}\log_\psi\left(w - e^{-\pi bi/a}\right) + (D + Ei).$$

The complex electrostatic potential between the parallel lines in the z-plane is

$$F(z) = V_0 - \frac{q}{2\pi\epsilon_0}\log_\phi\left(e^{\pi z/a} - e^{\pi bi/a}\right) + \frac{q}{2\pi\epsilon_0}\log_\psi\left(e^{\pi z/a} - e^{-\pi bi/a}\right) + (D + Ei).$$

The real potential is

$$\begin{aligned}
V(x,y) &= \mathrm{Re}\,[F(z)] = V_0 + \frac{q}{2\pi\epsilon_0}\mathrm{Re}\left[\log_\psi\left(e^{\pi z/a} - e^{-\pi bi/a}\right) - \log_\phi\left(e^{\pi z/a} - e^{\pi bi/a}\right)\right] + D \\
&= V_0 + \frac{q}{2\pi\epsilon_0}\mathrm{Re}\left[\log_\psi\left(e^{\pi(x+yi)/a} - e^{-\pi bi/a}\right) - \log_\phi\left(e^{\pi(x+yi)/a} - e^{\pi bi/a}\right)\right] + D \\
&= V_0 + \frac{q}{2\pi\epsilon_0}\mathrm{Re}\left\{\log_\psi\left[\left(e^{\pi x/a}\cos\frac{\pi y}{a} - \cos\frac{\pi b}{a}\right) + \left(e^{\pi x/a}\sin\frac{\pi y}{a} + \sin\frac{\pi b}{a}\right)i\right]\right. \\
&\qquad \left. - \log_\phi\left[\left(e^{\pi x/a}\cos\frac{\pi y}{a} - \cos\frac{\pi b}{a}\right) + \left(e^{\pi x/a}\sin\frac{\pi y}{a} - \sin\frac{\pi b}{a}\right)i\right]\right\} + D \\
&= V_0 + \frac{q}{2\pi\epsilon_0}\left\{\ln\sqrt{\left(e^{\pi x/a}\cos\frac{\pi y}{a} - \cos\frac{\pi b}{a}\right)^2 + \left(e^{\pi x/a}\sin\frac{\pi y}{a} + \sin\frac{\pi b}{a}\right)^2}\right. \\
&\qquad \left. - \ln\sqrt{\left(e^{\pi x/a}\cos\frac{\pi y}{a} - \cos\frac{\pi b}{a}\right)^2 + \left(e^{\pi x/a}\sin\frac{\pi y}{a} - \sin\frac{\pi b}{a}\right)^2}\right\} + D \\
&= V_0 + \frac{q}{4\pi\epsilon_0}\ln\left[\frac{\left(e^{\pi x/a}\cos\frac{\pi y}{a} - \cos\frac{\pi b}{a}\right)^2 + \left(e^{\pi x/a}\sin\frac{\pi y}{a} + \sin\frac{\pi b}{a}\right)^2}{\left(e^{\pi x/a}\cos\frac{\pi y}{a} - \cos\frac{\pi b}{a}\right)^2 + \left(e^{\pi x/a}\sin\frac{\pi y}{a} - \sin\frac{\pi b}{a}\right)^2}\right] + D \\
&= V_0 + \frac{q}{4\pi\epsilon_0}\ln\left[\frac{e^{2\pi x/a} + 1 + 2e^{\pi x/a}\left(\sin\frac{\pi y}{a}\sin\frac{\pi b}{a} - \cos\frac{\pi y}{a}\cos\frac{\pi b}{a}\right)}{e^{2\pi x/a} + 1 - 2e^{\pi x/a}\left(\cos\frac{\pi y}{a}\cos\frac{\pi b}{a} + \sin\frac{\pi y}{a}\sin\frac{\pi b}{a}\right)}\right] + D \\
&= V_0 + \frac{q}{4\pi\epsilon_0}\ln\left[\frac{e^{2\pi x/a} + 1 - 2e^{\pi x/a}\cos\frac{\pi(y+b)}{a}}{e^{2\pi x/a} + 1 - 2e^{\pi x/a}\cos\frac{\pi(y-b)}{a}}\right] + D \\
&= V_0 + \frac{q}{4\pi\epsilon_0}\ln\left[\frac{\cosh\frac{\pi x}{a} - \cos\frac{\pi(y+b)}{a}}{\cosh\frac{\pi x}{a} - \cos\frac{\pi(y-b)}{a}}\right] + D.
\end{aligned}$$

Either of the conditions that $V = V_0$ along $y = 0$ or $y = a$ leads to $D = 0$. Equipotentials are defined implicitly by the equation

$$C = V_0 + \frac{q}{4\pi\epsilon_0}\ln\left[\frac{\cosh\dfrac{\pi x}{a} - \cos\dfrac{\pi(y+b)}{a}}{\cosh\dfrac{\pi x}{a} - \cos\dfrac{\pi(y-b)}{a}}\right].$$

They are shown in Figure 7.82.●

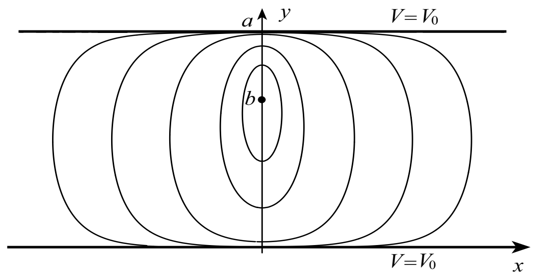


Figure 7.82

EXERCISES 7.8.2

1. The positive x-axis and a half-line from the origin at a positive rotation of angle α are both held at potential V_0. A line charge of q coulombs per metre, perpendicular to the xy-plane, is at the point $z_0 = Re^{\phi i}$ between the lines. Show that the electrostatic potential between the lines is

$$V(r,\theta) = V_0 + \frac{q}{4\pi\epsilon_0}\ln\left[\frac{r^{2\pi/\alpha} + R^{2\pi/\alpha} - 2r^{\pi/\alpha}R^{\pi/\alpha}\cos\dfrac{\pi(\theta+\phi)}{\alpha}}{r^{2\pi/\alpha} + R^{2\pi/\alpha} - 2r^{\pi/\alpha}R^{\pi/\alpha}\cos\dfrac{\pi(\theta-\phi)}{\alpha}}\right].$$

2. A line charge of q coulombs per metre of length is located at the point $x = a$ in the xy-plane. Surrounding the charge is a circle of radius R, centred at the origin, that is held at potential V_0.

(a) Find the potential inside the circle by mapping the situation to a circle with the line charge at the centre of the circle.

(b) Show that if a negative line charge is placed at the inverse point of the positive one in the circle (see equation 7.11 in Section 7.2), then their combination leads to the required potential. This is an example of the method of images where the image is the inverse point in a circle.

3. A line charge of q coulombs per metre is at $x = a$ on the x-axis. A conducting circle of radius R ($R < a$), centred at the origin, is held at potential V_0. Show that the electrostatic potential outside the circle is

$$V(x,y) = V_0 + \frac{q}{4\pi\epsilon_0}\ln\left\{\frac{(ax-R^2)^2 + a^2y^2}{R^2[(x-a)^2+y^2]}\right\}.$$

4. A semi-infinite channel consists of the region bounded by the x-axis, $-a \le x \le a$, and the lines $x = \pm a$, $y \ge 0$. Each of the lines is at potential V_0. A line charge of q coulombs per metre perpendicular to the xy-plane is located at the point $(0, b)$. Show that the potential in the channel is

$$V(x,y) = V_0 + \frac{q}{4\pi\epsilon_0} \ln \left(\frac{\sin^2 \frac{\pi x}{2a} + \sinh^2 \frac{\pi y}{2a} + \sinh^2 \frac{\pi b}{2a} + 2\cos \frac{\pi x}{2a} \sinh \frac{\pi y}{2a} \sinh \frac{\pi b}{2a}}{\sin^2 \frac{\pi x}{2a} + \sinh^2 \frac{\pi y}{2a} + \sinh^2 \frac{\pi b}{2a} - 2\cos \frac{\pi x}{2a} \sinh \frac{\pi y}{2a} \sinh \frac{\pi b}{2a}} \right).$$

5. The positive x- and y-axes have potential V_0. A line charge of q coulombs per metre, at the point (x_0, y_0), is perpendicular to the xy-plane.
(a) Use the method of images with an equal line charge at $(-x_0, -y_0)$ and negative line charges of the same strength at the points $(-x_0, y_0)$ and $(x_0, -y_0)$ to show that potential in the first quadrant can be expresed in the form

$$V(x,y) = V_0 + \frac{q}{4\pi\epsilon_0} \ln \left\{ \frac{[(x-x_0)^2 + (y+y_0)^2][(x+x_0)^2 + (y-y_0)^2]}{[(x-x_0)^2 + (y-y_0)^2][(x+x_0)^2 + (y+y_0)^2]} \right\}.$$

(b) Use the mapping $w = z^2$ and Example 7.37 to write the potential in the form

$$V(r,\theta) = V_0 + \frac{q}{4\pi\epsilon_0} \ln \left\{ \frac{r^4 + R^4 - 2r^2R^2 \cos[2(\theta+\phi)]}{r^4 + R^4 - 2r^2R^2 \cos[2(\theta-\phi)]} \right\},$$

where $z_0 = Re^{\phi i}$.
(c) Show that the results in parts (a) and (b) are the same.

6. The semicircle $x^2 + y^2 = R^2$, $y > 0$ and the x-axis for $|x| \geq R$ are at potential V_0. A line charge of q coulombs per metre perpendicular to the xy-plane is at the point $z_0 = ae^{\phi i}$, where $a > R$. Show that the potential in the half-plane $y > 0$ outside the semicircle is

$$V(r,\theta) = V_0 + \frac{q}{4\pi\epsilon_0} \ln \left\{ \frac{[r^2 + a^2 - 2ar\cos(\theta+\phi)][a^2R^2 + 1 - 2ar\cos(\theta-\phi)]}{[r^2 + a^2 - 2ar\cos(\theta-\phi)][a^2R^2 + 1 - 2ar\cos(\theta+\phi)]} \right\}.$$

Hint: Map the region of interest to the half-plane $v > 0$ with $w = z + 1/z$ and use Example 7.37.

7. The positive x-axis is held at potential V_0, and there is a line of charge q coulombs per metre, perpendicular to the xy-plane, at the point $z_0 = Re^{\phi i}$. Use Exercise 1 to find electrostatic potential in the xy-plane.