§6.2 Evaluation of Definite Integrals

We have used definite integrals to evaluate contour integrals. It may come as a surprise to learn that contour integrals and residues can be used to evaluate certain classes of definite integrals that might otherwise prove intractable.

Definite Integrals Involving Trigonometric Functions

Contour integrals and residues can be useful in the evaluation of definite integrals of the form

$$\int_a^b \frac{P(\cos\theta, \sin\theta)}{Q(\cos\theta, \sin\theta)}\, d\theta \tag{6.4}$$

where $P(\cos\theta, \sin\theta)$ and $Q(\cos\theta, \sin\theta)$ are polynomials in $\cos\theta$ and $\sin\theta$, provided Q is never equal to zero.

Example 6.6 Evaluate $\displaystyle\int_0^{2\pi} \frac{1}{2-\cos\theta}\, d\theta.$

Solution We transform the definite integral to the complex plane by setting $z = e^{\theta i}$ and $dz = ie^{\theta i} d\theta = iz\, d\theta$. As θ traces out the values 0 through 2π, z traces out the circle $C : |z| = 1$ once counterclockwise (Figure 6.5). We replace $\cos\theta$ with $\cos\theta = (e^{\theta i} + e^{-\theta i})/2 = (z + z^{-1})/2$,

$$\int_0^{2\pi} \frac{1}{2-\cos\theta}\, d\theta = \oint_C \frac{1}{2 - \left(\frac{z+z^{-1}}{2}\right)} \frac{dz}{iz} = 2i \oint_C \frac{1}{z^2 - 4z + 1}\, dz.$$

The (real) definite integral has been replaced by a (complex) contour integral. Because $z^2 - 4z + 1 = 0$ when $z = (4 \pm \sqrt{16-4})/2 = 2 \pm \sqrt{3}$, the integrand

$$\frac{1}{z^2 - 4z + 1} = \frac{1}{(z - 2 - \sqrt{3})(z - 2 + \sqrt{3})}$$

has simple poles at $z = 2 \pm \sqrt{3}$, only one of which is interior to C. Since

$$\operatorname{Res}\left[\frac{1}{z^2 - 4z + 1}, 2 - \sqrt{3}\right] = \lim_{z \to 2-\sqrt{3}} \frac{z - 2 + \sqrt{3}}{(z - 2 - \sqrt{3})(z - 2 + \sqrt{3})} = -\frac{1}{2\sqrt{3}},$$

Cauchy's residue theorem gives

$$\int_0^{2\pi} \frac{1}{2-\cos\theta}\, d\theta = 2i\left[2\pi i\left(-\frac{1}{2\sqrt{3}}\right)\right] = \frac{2\pi}{\sqrt{3}}. \bullet$$

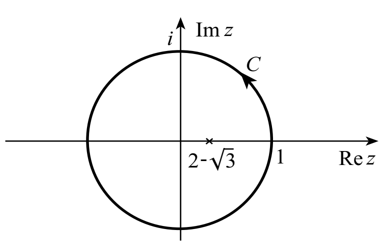


Figure 6.5

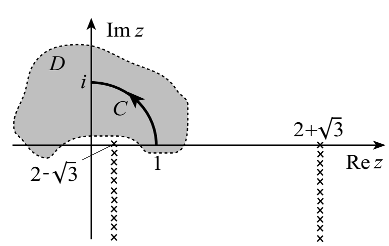


Figure 6.6

This method can be applied to evaluate integrals over intervals other than $[0, 2\pi]$, but residue theory may not be applicable.

Example 6.7 Evaluate $\displaystyle\int_0^{\pi/2} \frac{1}{2-\cos\theta}\,d\theta.$

Solution The transformation $z = e^{\theta i}$ transforms the real integral into the contour integral of Example 6.6 over the quarter circle in Figure 6.6. This time we use the partial fraction decomposition of the integrand to write

$$\int_0^{\pi/2} \frac{1}{2-\cos\theta}\,d\theta = 2i\int_C \frac{1}{z^2-4z+1}\,dz = 2i\int_C \left(\frac{\frac{1}{2\sqrt{3}}}{z-2-\sqrt{3}} + \frac{\frac{-1}{2\sqrt{3}}}{z-2+\sqrt{3}}\right)dz.$$

In the domain D of Figure 6.6, the integrand has an antiderivative, and hence

$$\int_0^{\pi/2} \frac{1}{2-\cos\theta}\,d\theta = \frac{i}{\sqrt{3}}\left\{\log_\phi(z-2-\sqrt{3}) - \log_\phi(z-2+\sqrt{3})\right\}_1^i,$$

where we choose branches of the logarithm functions with branch cuts $\phi = -\pi/2$. Then

$$\begin{aligned}\int_0^{\pi/2} \frac{1}{2-\cos\theta}\,d\theta &= \frac{i}{\sqrt{3}}\Big[\log_{-\pi/2}(i-2-\sqrt{3}) - \log_{-\pi/2}(i-2+\sqrt{3}) \\ &\qquad - \log_{-\pi/2}(-1-\sqrt{3}) + \log_{-\pi/2}(-1+\sqrt{3})\Big] \\ &= \frac{i}{\sqrt{3}}\left\{\left[\ln\sqrt{(-2-\sqrt{3})^2+1} + i\left(\pi - \mathrm{Tan}^{-1}\left(\frac{1}{2+\sqrt{3}}\right)\right)\right]\right. \\ &\qquad - \left[\ln\sqrt{(-2+\sqrt{3})^2+1} + i\left(\pi - \mathrm{Tan}^{-1}\left(\frac{1}{2-\sqrt{3}}\right)\right)\right] \\ &\qquad \left. - [\ln(1+\sqrt{3}) + \pi i] + \ln(\sqrt{3}-1)\right\} \\ &= \frac{i}{\sqrt{3}}\left\{\left[\ln\sqrt{8+4\sqrt{3}} - \ln\sqrt{8-4\sqrt{3}} - \ln(1+\sqrt{3}) + \ln(\sqrt{3}-1)\right]\right. \\ &\qquad \left. + i\left[-\pi + \mathrm{Tan}^{-1}\left(\frac{1}{2-\sqrt{3}}\right) - \mathrm{Tan}^{-1}\left(\frac{1}{2+\sqrt{3}}\right)\right]\right\}.\end{aligned}$$

We can bring the inverse tangents together using the following identity. When $AB > 0$,

$$\mathrm{Tan}^{-1}A - \mathrm{Tan}^{-1}B = \mathrm{Tan}^{-1}\left(\frac{A-B}{1+AB}\right).$$

The result is

$$\begin{aligned}\int_0^{\pi/2} \frac{1}{2-\cos\theta}\,d\theta &= \frac{i}{2\sqrt{3}}\ln\left[\frac{8+4\sqrt{3}}{8-4\sqrt{3}}\frac{(\sqrt{3}-1)^2}{(\sqrt{3}+1)^2}\right] \\ &\qquad - \frac{1}{\sqrt{3}}\left\{-\pi + \mathrm{Tan}^{-1}\left[\frac{\frac{1}{2-\sqrt{3}} - \frac{1}{2+\sqrt{3}}}{1+\left(\frac{1}{2-\sqrt{3}}\right)\left(\frac{1}{2+\sqrt{3}}\right)}\right]\right\}\end{aligned}$$

$$= \frac{i}{2\sqrt{3}} \ln \left[\frac{8+4\sqrt{3}}{8-4\sqrt{3}} \frac{4-2\sqrt{3}}{4+2\sqrt{3}} \right] - \frac{1}{\sqrt{3}} \left[-\pi + \text{Tan}^{-1}\sqrt{3}) \right]$$

$$= \frac{2\sqrt{3}\pi}{9}. \bullet$$

Real Improper Integrals

Residues can also be effective in evaluation of improper integrals which have infinite upper or lower limits.

Example 6.8 Evaluate $\displaystyle\int_{-\infty}^{\infty} \frac{1}{1+x^4}\,dx.$

Solution It is fairly clear that were we to evaluate the contour integral

$$\oint_C \frac{1}{1+z^4}\,dz$$

where C is shown in Figure 6.7a, and were we to let $R \to \infty$, then that part of the contour integral along the real axis would give rise to the required improper integral. Let us consider this contour integral then.

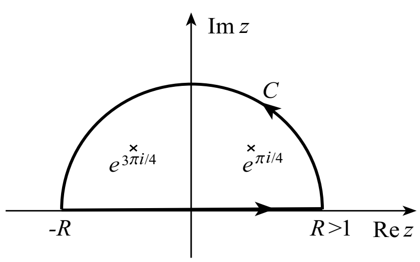


Figure 6.7a

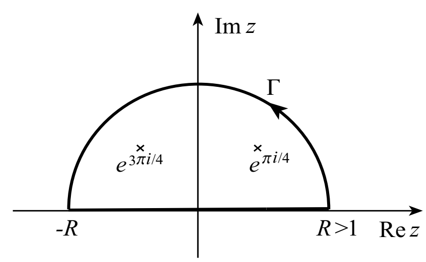


Figure 6.7b

The integrand $(1+z^4)^{-1}$ has simple poles at the four fourth roots of -1,

$$e^{\pi i/4}, \qquad e^{3\pi i/4}, \qquad e^{5\pi i/4}, \qquad e^{7\pi i/4},$$

only the first two of which are interior to C. L'Hôpital's rule (Theorem 5.24) gives

$$\text{Res}\left[\frac{1}{1+z^4}, e^{\pi i/4}\right] = \lim_{z\to e^{\pi i/4}} \frac{z-e^{\pi i/4}}{1+z^4} = \lim_{z\to e^{\pi i/4}} \frac{1}{4z^3} = \frac{1}{4e^{3\pi i/4}} = -\frac{\sqrt{2}}{8}(1+i).$$

Similarly, $\text{Res}\left[\dfrac{1}{1+z^4}, e^{3\pi i/4}\right] = \dfrac{\sqrt{2}}{8}(1-i)$. By Cauchy's residue theorem then,

$$\oint_C \frac{1}{1+z^4}\,dz = 2\pi i\left[-\frac{\sqrt{2}}{8}(1+i) + \frac{\sqrt{2}}{8}(1-i)\right] = \frac{\pi}{\sqrt{2}}.$$

Suppose we now divide C into a semicircular part Γ and a straight line part (Figure 6.7b). Then

$$\frac{\pi}{\sqrt{2}} = \int_{-R}^{R} \frac{1}{1+x^4}\,dx + \int_\Gamma \frac{1}{1+z^4}\,dz.$$

If we set $z = Re^{\theta i}$, $0 \le \theta \le \pi$, on Γ, then inequality 1.39 on the semicircle gives

$$\left|\frac{1}{1+z^4}\right| \le \frac{1}{|z^4|-1} = \frac{1}{R^4-1}.$$

Hence, by property 4.21,

$$\left|\int_\Gamma \frac{1}{1+z^4}\,dz\right| \le \frac{1}{R^4-1}(\pi R).$$

It is clear that the limit of this expression is zero as $R \to \infty$, and therefore

$$\frac{\pi}{\sqrt{2}} = \lim_{R\to\infty}\left(\int_{-R}^{R} \frac{1}{1+x^4}\,dx + \int_\Gamma \frac{1}{1+z^4}\,dz\right) = \int_{-\infty}^{\infty} \frac{1}{1+x^4}\,dx.\bullet$$

This example has illustrated that the contour integral of $1/(1+z^4)$ around the curve C of Figure 6.7 can be used to evaluate the improper integral of $1/(1+x^4)$ from negative infinity to infinity. The real difficulty in such problems is the choice of contour and the choice of integrand. We now do two more examples to illustrate some of these difficulties.

Example 6.9 Evaluate $\displaystyle\int_0^\infty \frac{\cos x}{x^2+1}\,dx.$

Solution We might consider the contour integral

$$\oint_C \frac{\cos z}{z^2+1}\,dz$$

around the contour in Figure 6.7. Certainly along the straight line portion of C the integrand reduces to $\cos x/(x^2+1)$, but if we set $z = x + yi$ along Γ, then on the semicircle

$$\left|\frac{\cos z}{z^2+1}\right| = \left|\frac{e^{(x+yi)i} + e^{-(x+yi)i}}{2(z^2+1)}\right| = \left|\frac{e^{-y+xi} + e^{y-xi}}{2(z^2+1)}\right|$$

which becomes infinite as $|z| \to \infty$. We shall not therefore be able to show that the contour integral along Γ approaches zero as $|z| \to \infty$ as in Example 6.8. This means that the choice of $\cos z/(z^2+1)$ was perhaps not a convenient one. Consider instead

$$\oint_C \frac{e^{zi}}{z^2+1}\,dz$$

where C is again the contour in Figure 6.7. Since the integrand has simple poles at $z = \pm i$, only $z = i$ being interior to C,

$$\oint_C \frac{e^{zi}}{z^2+1}\,dz = 2\pi i\,\mathrm{Res}\left[\frac{e^{zi}}{z^2+1}, i\right] = 2\pi i \lim_{z\to i}\left[\frac{(z-i)e^{zi}}{(z-i)(z+i)}\right] = 2\pi i\frac{e^{i(i)}}{2i} = \frac{\pi}{e}.$$

Thus,

$$\begin{aligned}\frac{\pi}{e} &= \int_{-R}^{R} \frac{e^{xi}}{x^2+1}\,dx + \int_\Gamma \frac{e^{zi}}{z^2+1}\,dz \\ &= \int_{-R}^{R} \frac{\cos x}{x^2+1}\,dx + i\int_{-R}^{R} \frac{\sin x}{x^2+1}\,dx + \int_\Gamma \frac{e^{zi}}{z^2+1}\,dz\end{aligned}$$

$$= \int_{-R}^{R} \frac{\cos x}{x^2+1}\,dx + \int_\Gamma \frac{e^{zi}}{z^2+1}\,dz \quad \text{(since } \sin x/(x^2+1) \text{ is an odd function).}$$

If we set $z = x + yi = Re^{\theta i}$ on Γ, then on this semicircle,

$$\left|\frac{e^{zi}}{z^2+1}\right| \le \frac{|e^{(x+yi)i}|}{|z^2|-1} \qquad \text{(by inequality 1.39)}$$
$$= \frac{e^{-y}}{R^2-1}$$
$$\le \frac{1}{R^2-1} \qquad \text{(since } y \ge 0).$$

Hence, by inequality 4.21,

$$\left|\int_\Gamma \frac{e^{zi}}{z^2+1}\,dz\right| \le \frac{1}{R^2-1}(\pi R).$$

Since the limit of this expression is zero as $R \to \infty$, it follows that

$$\frac{\pi}{e} = \lim_{R\to\infty}\left(\int_{-R}^{R} \frac{\cos x}{x^2+1}\,dx + \int_\Gamma \frac{e^{zi}}{z^2+1}\,dz\right) = \int_{-\infty}^{\infty} \frac{\cos x}{x^2+1}\,dx = 2\int_0^\infty \frac{\cos x}{x^2+1}\,dx.$$

Finally,

$$\int_0^\infty \frac{\cos x}{x^2+1}\,dx = \frac{\pi}{2e}. \bullet$$

This example illustrated that we do not always replace x's by z's to obtain the appropriate contour integral. The following example indicates that the choice of contour may not always be obvious.

Example 6.10 Evaluate $\displaystyle\int_0^\infty \frac{1}{1+x^3}\,dx.$

Solution Based on Example 6.8 we should perhaps consider

$$\oint_C \frac{1}{1+z^3}\,dz$$

where C is some appropriate contour. Clearly a part of C should be the positive real axis and possibly a circular arc of radius $R > 1$. But we cannot take a semicircle as in Example 6.8 since $1/(1+z^3)$ has a singularity at $z = -1$ on the negative real axis. What we should like to do then is choose some other line eminating from the origin say $z = re^{\phi i}$, $0 < \phi < \pi$, which leads to a simple solution (Figure 6.8). The integrand has simple poles at $z = e^{\pi i/3}$, -1, $e^{5\pi i/3}$.

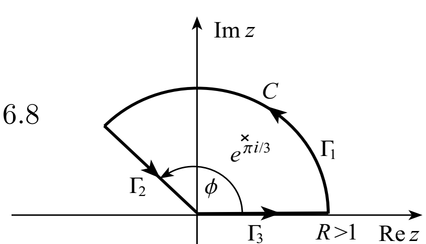


Figure 6.8

Suppose for the moment we stipulate that ϕ be in the interval $\pi/3 < \phi < \pi$, but, for the moment, leave it otherwise arbitrary. Cauchy's residue theorem and L'Hôpital's rule give

$$\oint_C \frac{1}{1+z^3}\,dz = 2\pi i \operatorname{Res}\left[\frac{1}{1+z^3}, e^{\pi i/3}\right] = 2\pi i \lim_{z\to e^{\pi i/3}} \frac{z - e^{\pi i/3}}{1+z^3}$$

$$= 2\pi i \lim_{z\to e^{\pi i/3}} \frac{1}{3z^2} = \frac{2\pi i}{3e^{2\pi i/3}} = \frac{(\sqrt{3}-i)\pi}{3}.$$

Thus,

$$\frac{(\sqrt{3}-i)\pi}{3} = \int_{\Gamma_1} \frac{1}{1+z^3}\,dz + \int_{\Gamma_2} \frac{1}{1+z^3}\,dz + \int_{\Gamma_3} \frac{1}{1+z^3}\,dz.$$

If we set $z = Re^{\theta i}$ on Γ_1, then on this arc

$$\left|\frac{1}{1+z^3}\right| \le \frac{1}{|z|^3-1} = \frac{1}{R^3-1},$$

and hence

$$\left|\int_{\Gamma_1} \frac{1}{1+z^3}\,dz\right| \le \frac{1}{R^3-1}(R\phi).$$

The limit of this expression is zero as $R \to \infty$. Thus,

$$\begin{aligned}\frac{(\sqrt{3}-i)\pi}{3} &= \lim_{R\to\infty}\left(\int_{\Gamma_1} \frac{1}{1+z^3}\,dz + \int_{\Gamma_2} \frac{1}{1+z^3}\,dz + \int_{\Gamma_3} \frac{1}{1+z^3}\,dz\right)\\ &= \int_{\Gamma_2} \frac{1}{1+z^3}\,dz + \int_0^\infty \frac{1}{1+x^3}\,dx,\end{aligned}$$

where $\Gamma_2 : z = re^{\phi i}$, $\infty > r \ge 0$ (and $\pi/3 < \phi < \pi$). Our problem then is to choose ϕ in order that the contour integral along Γ_2 can be evaluated. Since $z = re^{\phi i}$ on Γ_2,

$$\int_{\Gamma_2} \frac{1}{1+z^3}\,dz = \int_\infty^0 \frac{1}{1+r^3e^{3\phi i}}\,e^{\phi i}\,dr = -e^{\phi i}\int_0^\infty \frac{1}{1+r^3e^{3\phi i}}\,dr.$$

If we choose $\phi = 2\pi/3$, then

$$\int_{\Gamma_2} \frac{1}{1+z^3}\,dz = -e^{2\pi i/3}\int_0^\infty \frac{1}{1+r^3}\,dr,$$

and

$$\begin{aligned}\frac{(\sqrt{3}-i)\pi}{3} &= -e^{2\pi i/3}\int_0^\infty \frac{1}{1+r^3}\,dr + \int_0^\infty \frac{1}{1+x^3}\,dx\\ &= \left[-\left(-\frac{1}{2}+\frac{\sqrt{3}i}{2}\right)+1\right]\int_0^\infty \frac{1}{1+x^3}\,dx.\end{aligned}$$

Thus,

$$\int_0^\infty \frac{1}{1+x^3}\,dx = \frac{(\sqrt{3}-i)\pi}{3}\,\frac{2}{3-\sqrt{3}i} = \frac{2\pi}{3\sqrt{3}}.\bullet$$

General results concerning improper integrals of the types in Examples 6.8–6.10 are discussed in Exercises 27 and 30.

EXERCISES 6.2

In Exercises 1–17 use a contour integral to evaluate the definite integral.

1. $\displaystyle\int_0^{2\pi} \frac{1}{3-\sin\theta}\,d\theta$

2. $\displaystyle\int_0^{2\pi} \frac{1}{3+2\cos\theta}\,d\theta$

3. $\displaystyle\int_0^{2\pi} \frac{1}{6+5\sin\theta}\,d\theta$

4. $\displaystyle\int_0^{2\pi} \frac{1}{\sin\theta+2\cos\theta+4}\,d\theta$

5. $\displaystyle\int_0^{2\pi} \frac{1}{4\cos^2\theta+3}\,d\theta$

6. $\displaystyle\int_0^{2\pi} \frac{\sin 2\theta}{2+\cos\theta}\,d\theta$

7. $\displaystyle\int_0^{2\pi} \frac{\sin^2\theta}{5+4\cos\theta}\,d\theta$

8. $\displaystyle\int_0^{2\pi} \frac{\cos\theta}{3+\cos\theta}\,d\theta$

9. $\displaystyle\int_0^{\pi} \frac{1}{3+2\cos\theta}\,d\theta$

10. $\displaystyle\int_0^{\pi/2} \frac{1}{6+5\sin\theta}\,d\theta$

11. $\displaystyle\int_0^{\pi/2} \frac{1}{3-\sin\theta}\,d\theta$

12. $\displaystyle\int_{-\pi/2}^{0} \frac{1}{5+2\cos\theta}\,d\theta$

13. $\displaystyle\int_0^{2} \frac{1}{3+2\sin\theta}\,d\theta$

14. $\displaystyle\int_0^{1} \frac{1}{4+3\cos\theta}\,d\theta$

15. $\displaystyle\int_{-\pi}^{0} \frac{1}{4+\cos\theta}\,d\theta$

16. $\displaystyle\int_{-\pi}^{\pi/2} \frac{1}{5+2\sin\theta}\,d\theta$

17. $\displaystyle\int_0^{\pi/2} \frac{\cos\theta}{3+\cos\theta}\,d\theta$

In Exercises 18–23 use a contour integral to evaluate the improper integral.

18. $\displaystyle\int_0^{\infty} \frac{1}{2+x^2}\,dx$

19. $\displaystyle\int_{-\infty}^{\infty} \frac{x^2}{(x^2+1)(x^2+4)}\,dx$

20. $\displaystyle\int_{-\infty}^{\infty} \frac{1}{3+x^6}\,dx$

21. $\displaystyle\int_0^{\infty} \frac{x}{1+x^3}\,dx$

22. $\displaystyle\int_{-\infty}^{\infty} \frac{\sin x}{x^2+2x+2}\,dx$

23. $\displaystyle\int_0^{\infty} \frac{x^2\cos x}{(x^2+9)^2}\,dx$

24. Show that $\displaystyle\int_0^{2\pi} \cos^{2n}\theta\,d\theta = \frac{(2n)!\pi}{2^{2n-1}(n!)^2}$.

25. (a) Use the substitution $u = 1/x$ to show that

$$I = \int_0^1 \frac{1+x^2}{1+x^4}\,dx = \int_1^{\infty} \frac{1+u^2}{1+u^4}\,du,$$

and hence

$$I = \frac{1}{4}\int_{-\infty}^{\infty} \frac{1+x^2}{1+x^4}\,dx.$$

(b) Now use contour integration to calculate I.

26. Use contour integrals to prove the result of Exercise 3 in Section 4.8.

27. Example 6.8 and Exercises 18–21 are examples of improper integrals of the form

$$\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)}\, dx$$

where $P(x)$ and $Q(x)$ are polynomials (of degrees m and n), and $Q(x) \neq 0$ for all real x. Show that when $n \geq m + 2$,

$$\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)}\, dx = 2\pi i \left\{ \begin{array}{l} \text{sum of the residues of } P(z)/Q(z) \text{ at} \\ \text{its poles in the half-plane } \operatorname{Im} z > 0 \end{array} \right\}.$$

With this result, it is no longer necessary to introduce the contour of Figure 6.7. The fact that $n \geq m + 2$ guarantees that the contour integral along Γ vanishes as $R \to \infty$.

Use Exercise 27 to evaluate the improper integral in Exercises 28–29.

28. $\displaystyle\int_{-\infty}^{\infty} \frac{1}{(x^2 + 4x + 5)^2}\, dx$

29. $\displaystyle\int_{-\infty}^{\infty} \frac{x^2 + 3}{(x^2 + 1)(x^2 - x + 1)}\, dx$

30. Example 6.9 and Exercises 22 and 23 are examples of improper integrals of the form

$$\int_{-\infty}^{\infty} \frac{P(x)\cos ax}{Q(x)}\, dx \qquad \text{or} \qquad \int_{-\infty}^{\infty} \frac{P(x)\sin ax}{Q(x)}\, dx$$

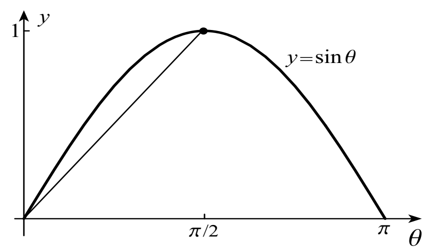
where $P(x)$ and $Q(x)$ are polynomials (of degrees m and n), $a > 0$ is real, and $Q(x) \neq 0$ for any real x.

(a) Use the figure to the right to verify that

$$\sin\theta \geq \frac{2\theta}{\pi}, \quad 0 \leq \theta \leq \pi/2,$$

called Jordan's inequality. Prove that for $a > 0$,

$$\int_0^{\pi} e^{-aR\sin\theta}\, d\theta \leq \frac{\pi}{aR}(1 - e^{-aR}).$$



(b) Show that when $n \geq m + 1$, and $a > 0$,

$$\int_{-\infty}^{\infty} \frac{P(x)\cos ax}{Q(x)}\, dx = -2\pi \operatorname{Im} \left\{ \begin{array}{l} \text{sum of the residues of } P(z)e^{azi}/Q(z) \\ \text{at its poles in the half-plane } \operatorname{Im} z > 0 \end{array} \right\}$$

and

$$\int_{-\infty}^{\infty} \frac{P(x)\sin ax}{Q(x)}\, dx = 2\pi \operatorname{Re} \left\{ \begin{array}{l} \text{sum of the residues of } P(z)e^{azi}/Q(z) \\ \text{at its poles in the half-plane } \operatorname{Im} z > 0 \end{array} \right\}.$$

Hint: Use the contour in Figure 6.7, and show that R can be chosen sufficiently large that on Γ,

$$\left| \frac{P(z)e^{azi}}{Q(z)} \right| \leq \frac{(|a_m|R^m + \cdots + |a_0|)e^{-aR\sin\theta}}{|b_n|R^n - \cdots - |b_0|},$$

where $P(z) = a_m z^m + \cdots + a_0$ and $Q(z) = b_n z^n + \cdots + b_0$. Now use the result in part (a).

In Exercises 31–35 use the result in Exercise 30 to evaluate the improper integral.

31. $\displaystyle\int_0^{\infty} \frac{x\sin 2x}{x^2 + 5}\, dx$

32. $\displaystyle\int_{-\infty}^{\infty} \frac{x^2\cos 3x}{x^4 + 4}\, dx$

33. $\displaystyle\int_0^{\infty} \frac{\sin^2 x}{x^2 + 1}\, dx$

34. $\displaystyle\int_{-\infty}^{\infty} \frac{\cos x}{(x + a)^2 + b^2}\, dx$

35. Evaluate $\displaystyle\int_0^{\pi/2} \frac{1}{(3-\sin\theta)^2}\,d\theta$.

In Exercises 36–38 verify the formula for the given values of the parameters.

36. $\displaystyle\int_0^{2\pi} \frac{1}{a+b\sin\theta}d\theta = \frac{2\pi}{\sqrt{a^2-b^2}}$, when $0<|b|<a$

37. $\displaystyle\int_0^{2\pi} \frac{1}{a+b\cos\theta}d\theta = \frac{2\pi}{\sqrt{a^2-b^2}}$, when $0<|b|<a$

38. $\displaystyle\int_0^{2\pi} \frac{1}{d+a\cos\theta+b\sin\theta}\,d\theta = \frac{2\pi\operatorname{sgn} d}{\sqrt{d^2-a^2-b^2}}$, where $\operatorname{sgn} d = \begin{cases} 1, & d>0 \\ -1, & d<0 \end{cases}$, when a, b, and d are real with $d^2>a^2+b^2$.