PREFACE

To the Instructor

This text is written for undergraduate students in science and engineering. At a brisk pace, theory and a selection of applications can be covered in one semester; for complete coverage at a more leisurely pace allow two semesters. The presentation assumes a working knowledge of single-variable real calculus, including infinite series, and familiarity with partial differentiation and line integrals. Discussions are rigorous, but the approach is intuitive.

Chapter 1 introduces complex numbers. We add, subtract, multiply, divide, and find roots of complex numbers. We introduce the Cartesian, polar, and exponential forms for complex numbers, illustrate complex numbers geometrically, and solve polynomial equations. Chapter 2 begins with the algebra of complex functions, discusses limits and continuity, and ends with complex differentiation and analyticity. The complex exponential, trigonometric, hyperbolic, logarithmic, and inverse trigonometric and hyperbolic functions are introduced in Chapter 3. We concentrate on algebraic properties of these functions, and their zeros and singularities, but, in preparation for conformal mapping in Chapter 7, we also stress the importance of visualizing a function $w = f(z)$ as a mapping from the z-plane to the w-plane. The fact that real and imaginary parts of analytic functions are harmonic suggests the use of complex functions to problems in electrostatics, steady-state temperature distributions, and fluid flows. These applications are introduced in Chapter 3.

In Chapter 4 we focus on contour integration. We verify the Cauchy-Goursat theorem, introduce Cauchy's integral formulas and Poisson's integral formulas, and develop modulus principles for complex and harmonic functions. Power series and Laurent series with the subsequent classification of singularities of complex functions are discussed in Chapter 5. Chapters 4 and 5 culminate in the theory of residues in Chapter 6. The power of residues is brought out in the evaluation of real trigonometric and improper integrals, the principle of the argument, Rouche's theorem, and summations of certain types of infinite series.

Although the first six chapters emphasize algebraic consequences of analyticity, and geometric results in the context of conformal mapping are discussed in Chapter 7, we constantly stress the geometric interpretation of a function as a mapping. Having students repeatedly visualize regional mappings by exponential, trigonometric, and hyperbolic functions, and their inverses, we pave the way for the application of conformal mappings to problems in heat flow, fluid flow, and electrostatics. These applications are dealt with in Chapter 7, perhaps with more detail than most texts at this level, but they are introduced in Chapter 3 as illustrations of the fact that families of curves defined by harmonic conjugates are orthogonal trajectories. In this way we give students an early indication that complex functions are indeed useful in physical applications.

In Chapter 8, we discuss Laplace transforms. We develop their elementary properties and show how they can be used to replace ordinary differential equations with algebraic equations. Cauchy's integral formula and residues then lead to a direct formula for inverting Laplace transforms. The transform is used to solve partial differential equations on bounded and semi-infinite domains. Finally, we introduce transfer functions and the notion of stability for systems governed by differential equations.

In Chapter 9, we use the Fourier transform, the Fourier sine transform, and the Fourier cosine transform to reduce partial differential equations on infinite and semi-infinite intervals to ordinary differential equations. We apply the technique to heat conduction, vibration, and potential problems.

In Chapter 10, we discuss the z-transform. The discussion parallels that for the Laplace transform in that we discuss properties of the transform, followed by its use in solving first- and second-order difference equations. Applications are discussed for difference equations as are transfer functions and stability for discrete systems.

We take every opportunity to compare properties of complex functions to those of real functions. Only then can the power and elegance of complex variable theory truly be appreciated. Plenty of exercises, with answers, are provided. Exercises are graded from routine, for reinforcement of fundamentals, to challenging, for the more enterprising reader. A complete solutions manual containing detailed solutions to all problems is available.

To the Student

When the domain and range of a function are sets of complex numbers, we speak of a complex function of a complex variable. This text is an introduction to the properties and applications of such functions; in particular, we develop the *calculus* of complex functions. Your earliest calculus studies involved real-valued functions $f(x)$ of a real variable x. You found their derivatives and integrals, and used these in many geometric and physical applications. Your second exposure to calculus expanded single-variable concepts to functions of many real variables $f(x, y, \ldots)$. Once again you differentiated and integrated these functions, and applied the derivatives and integrals in more complicated, but more realistic applications. It is now time to take the third, and for most, the final step; calculus of functions $f(z)$ of a complex variable z.

You are well aware of the connection between integration and differentiation in real calculus; you will be amazed at the intimate relationships among the three major topics of complex calculus — differentiation, integration, and infinite series. Properties in any one of these topics are reflected in properties in the other two, so much so, that we could commence complex calculus with any one of the three topics. Our preference is to begin with differentiation, it being, perhaps, the easiest of the three topics for most students in real calculus. We follow this with integration and infinite series.

Calculus of complex functions has many similarities to both single-variable (real) calculus and multivariable (real) calculus, and we shall point these out as discussions unfold. But there are also striking differences, and we shall be even more careful to draw these to your attention. For instance, in most applications of real calculus, we are concerned with points where functions are well-behaved. Contrarily, points where complex functions misbehave are often the most valuable in applications. Existence of the first derivative of a real function implies nothing about existence of higher order derivatives, whereas existence of the first derivative of a complex function implies existence of derivatives of all orders.

Complex calculus provides proofs to many results that seem otherwise intractable in real analysis; it also provides a clearer understanding to some topics in real analysis. For example, the often assumed result that every real polynomial has a zero is verified in Chapter 4; why there is a number called the "radius" of convergence of a real power series becomes clear when we study complex series in Chapter 5; and why solutions of Laplace's equation do not have relative extrema is verified in Chapter 4.

We use geometric visualizations to introduce and clarify ideas whenever possible. Visualizations of real functions $f(x)$ as curves and $f(x, y)$ as surfaces are unavailable for complex functions $f(z)$. Instead, we interpret $f(z)$ as a mapping from one complex plane to another. The mapping approach helps us appreciate many of the properties of functions, and in addition, paves the way for the important topic of conformal mapping in Chapter 7 with its applications to problems in electrostatics, heat flow, and fluid flow.

The study of calculus of complex functions can be a very rewarding experience. Topics unfold naturally, and each new topic intimately relates to everything that has gone before. Proofs of most results, even the very profound, are usually quite straightforward, and the material is rich in applications. To aid in recognizing when discussions are complete, we have designated the end of the proof of a theorem by ∎, and the end of an example by a •.

True understanding cannot be achieved simply by reading the text. You must engross yourself in the subject and we have included numerous exercises for this purpose. Try as many as you can; you will not regret the effort. Answers to exercises can be found in Appendix A. A solutions manual with detailed solutions to all problems is available.

Finally, we wish you every success in your studies, and we hope that you will share in our fascination with the subject.