§11.6 Application of Fourier Sine and Cosine Transforms to Initial Boundary Value Problems

Fourier sine and cosine transforms are used to solve initial boundary value problems associated with second order partial differential equations on the semi-infinite interval $x > 0$. Because property 11.43d for the Fourier sine transform utilizes the value of the function at $x = 0$, the sine transform is applied to problems with a Dirichlet boundary condition at $x = 0$. Similarly, property 11.43b indicates that the cosine transform should be used when the boundary condition at $x = 0$ is of Neumann type.

Example 11.20 Solve the vibration problem

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}, \quad x > 0, \quad t > 0, \tag{11.52a}$$

$$y(0,t) = f_1(t), \quad t > 0, \tag{11.52b}$$

$$y(x,0) = f(x), \quad x > 0, \tag{11.52c}$$

$$y_t(x,0) = g(x), \quad x > 0, \tag{11.52d}$$

for displacement of a semi-infinite string with prescribed motion at its left end $x = 0$.

Solution Because the boundary condition at $x = 0$ is Dirichlet, we apply the Fourier sine transform to the PDE and use property 11.43d for the transform of $\partial^2 y/\partial x^2$,

$$\frac{d^2\tilde{y}}{dt^2} = -\omega^2 c^2 \tilde{y}(\omega, t) + \omega c^2 f_1(t).$$

Thus, the Fourier sine transform $\tilde{y}(\omega, t)$ of $y(x,t)$ must satisfy the ODE

$$\frac{d^2\tilde{y}}{dt^2} + \omega^2 c^2 \tilde{y} = \omega c^2 f_1(t)$$

subject to transforms of initial conditions 11.52c,d,

$$\tilde{y}(\omega, 0) = \tilde{f}(\omega), \quad \tilde{y}'(\omega, 0) = \tilde{g}(\omega).$$

Variation of parameters leads to the following general solution of the ODE

$$\tilde{y}(\omega, t) = A\cos c\omega t + B \sin c\omega t + c\int_0^t f_1(u)\sin c\omega(t-u)\,du.$$

The initial conditions require A and B to satisfy

$$\tilde{f}(\omega) = A, \qquad \tilde{g}(\omega) = c\omega B.$$

Hence,

$$\tilde{y}(\omega, t) = \tilde{f}(\omega)\cos c\omega t + \frac{\tilde{g}(\omega)}{c\omega}\sin c\omega t + c\int_0^t f_1(u)\sin c\omega(t-u)\,du, \tag{11.53}$$

and $y(x,t)$ is the inverse transform of this function

$$y(x,t) = \frac{2}{\pi}\int_0^\infty \tilde{y}(\omega, t)\sin \omega x\, d\omega. \tag{11.54}$$

The first term in this integral is

$$\frac{2}{\pi}\int_0^\infty \tilde{f}(\omega)\cos c\omega t \sin \omega x\, d\omega = \frac{2}{\pi}\int_0^\infty \frac{1}{2}\tilde{f}(\omega)[\sin\omega(x-ct)+\sin\omega(x+ct)]\, d\omega$$
$$= \frac{1}{2}[f(x-ct)+f(x+ct)],$$

provided $f(x)$ is extended as an odd function.

According to Exercise 11 in Section 11.5, the Fourier cosine transform of $h(x+ct)-h(x-ct)$ is $(\sin c\omega t)/\omega$. Consequently, convolution identity 11.44d implies that the inverse sine transform of $[\tilde{g}(\omega)/(c\omega)]\sin c\omega t$ is

$$\frac{1}{2c}\int_0^\infty [h(v)-h(v-ct)][g(x+v)+g(x-v)]\, dv = \frac{1}{2c}\left[\int_0^{ct} g(x+v)\, dv + \int_0^{ct} g(x-v)\, dv\right],$$

provided $g(x)$ is extended as an odd function for $x<0$. When we set $u=x+v$ and $u=x-v$, respectively, in these integrals, the result is

$$\frac{1}{2c}\left[\int_x^{x+ct} g(u)\, du + \int_x^{x-ct} g(u)(-du)\right] = \frac{1}{2c}\int_{x-ct}^{x+ct} g(u)\, du.$$

The inverse transform of the integral term in $\tilde{y}(\omega,t)$ can also be expressed in closed form if we set $u=c(t-v)$,

$$c\int_0^t f_1(v)\sin c\omega(t-v)\, dv = c\int_{ct}^0 f_1\left(t-\frac{u}{c}\right)\sin\omega u\left(-\frac{du}{c}\right) = \int_0^{ct} f_1\left(t-\frac{u}{c}\right)\sin\omega u\, du.$$

But this is the Fourier sine transform of the function

$$\begin{cases} f_1\left(t-\dfrac{x}{c}\right), & x<ct \\ 0, & x>ct \end{cases}$$

or

$$\begin{cases} 0, & t<x/c \\ f_1\left(t-\dfrac{x}{c}\right), & t>x/c \end{cases} = f_1\left(t-\frac{x}{c}\right)h\left(t-\frac{x}{c}\right).$$

The solution is therefore

$$y(x,t) = \frac{1}{2}[f(x-ct)+f(x+ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(u)\, du + f_1\left(t-\frac{x}{c}\right)h\left(t-\frac{x}{c}\right). \quad (11.55)$$

The first two terms are the d'Alembert part of the solution. The last term is due to the nonhomogeneity at the end $x=0$; it can be interpreted physically, and this is most easily done when $f(x)=g(x)=0$. In this case, the complete solution is

$$y(x,t) = f_1\left(t-\frac{x}{c}\right)h\left(t-\frac{x}{c}\right).$$

A point x on the string remains at rest until time $t=x/c$, when it begins to execute the same motion as the end $x=0$. The time x/c taken by the disturbance to reach x is called **retarded time**. The disturbance $f_1(t)$ at $x=0$ travels down the string with velocity c.

The solution of the original problem is a superposition of the d'Alembert displacement and the displacement due to the end effect at $x=0$.●

Example 11.21 The temperature of a semi-infinite rod at time $t = 0$ is $f(x)$, $x \geq 0$. For time $t > 0$, heat is added to the rod uniformly over the end $x = 0$ at a variable rate $f_1(t)$ W/m^2. The initial boundary value problem for temperature $U(x,t)$ in the rod is

$$\frac{\partial U}{\partial t} = k\frac{\partial^2 U}{\partial x^2}, \quad x > 0, \quad t > 0, \tag{11.56a}$$

$$U_x(0,t) = -\kappa^{-1} f_1(t), \quad t > 0, \tag{11.56b}$$

$$U(x,0) = f(x), \quad x > 0. \tag{11.56c}$$

Find $U(x,t)$.

Solution Because the boundary condition at $x = 0$ is Neumann, we apply the Fourier cosine transform to the PDE and use property 11.43b,

$$\frac{d\tilde{U}}{dt} = -k\omega^2\tilde{U}(\omega,t) + \frac{k}{\kappa}f_1(t).$$

Thus, the Fourier cosine transform $\tilde{U}(\omega,t)$ must satisfy the ODE

$$\frac{d\tilde{U}}{dt} + k\omega^2\tilde{U} = \frac{k}{\kappa}f_1(t)$$

subject to the transform of initial condition 11.56c,

$$\tilde{U}(\omega,0) = \tilde{f}(\omega).$$

A general solution of the ODE is

$$\tilde{U}(\omega,t) = Ce^{-k\omega^2 t} + \frac{k}{\kappa}\int_0^t e^{-k\omega^2(t-u)}f_1(u)\,du,$$

and the initial condition requires $\tilde{f}(\omega) = C$. Consequently,

$$\tilde{U}(\omega,t) = \tilde{f}(\omega)e^{-k\omega^2 t} + \frac{k}{\kappa}\int_0^t e^{-k\omega^2(t-u)}f_1(u)\,du, \tag{11.57}$$

and the required temperature is the inverse cosine transform of this function. According to Exercise 9 in Section 11.5, the Fourier cosine transform of e^{-ax^2} is $\dfrac{1}{2}\sqrt{\dfrac{\pi}{a}}e^{-\omega^2/(4a)}$, or, conversely, the inverse Fourier cosine transform of $e^{-k\omega^2 t}$ is $\dfrac{1}{\sqrt{k\pi t}}e^{-x^2/(4kt)}$. Convolution property 11.44a therefore gives the inverse cosine transform of $\tilde{f}(\omega)e^{-k\omega^2 t}$ as

$$\frac{1}{2}\int_0^\infty f(v)\frac{1}{\sqrt{k\pi t}}[e^{-(x-v)^2/(4kt)} + e^{-(x+v)^2/(4kt)}]\,dv$$
$$= \frac{1}{2\sqrt{k\pi t}}\int_0^\infty f(v)[e^{-(x-v)^2/(4kt)} + e^{-(x+v)^2/(4kt)}]\,dv.$$

Furthermore, the inverse cosine transform of $e^{-k\omega^2(t-u)}$ is $\dfrac{1}{\sqrt{k\pi(t-u)}}e^{-x^2/[4k(t-u)]}$, and therefore the inverse transform of the integral term can be expressed in the form

$$\mathcal{F}_c^{-1}\left\{\int_0^t e^{-k\omega^2(t-u)}f_1(u)\,du\right\} = \int_0^t \frac{f_1(u)}{\sqrt{k\pi(t-u)}}e^{-x^2/[4k(t-u)]}\,du.$$

Thus, the temperature function is

$$U(x,t) = \frac{1}{2\sqrt{k\pi t}} \int_0^\infty f(v)[e^{-(x-v)^2/(4kt)} + e^{-(x+v)^2/(4kt)}]\, dv$$
$$+ \frac{\sqrt{k}}{\kappa\sqrt{\pi}} \int_0^t \frac{f_1(u)}{\sqrt{t-u}} e^{-x^2/[4k(t-u)]}\, du.\bullet \tag{11.58}$$

Example 11.22 Solve the following potential problem in the quarter plane $x > 0$, $y > 0$,

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0, \quad x > 0, \quad y > 0, \tag{11.59a}$$
$$V(0,y) = g(y), \quad y > 0, \tag{11.59b}$$
$$V_y(x,0) = f(x), \quad x > 0. \tag{11.59c}$$

Solution Superposition can be used to express $V(x,y)$ as the sum of functions $V_1(x,y)$ and $V_2(x,y)$ satisfying

$$\frac{\partial^2 V_1}{\partial x^2} + \frac{\partial^2 V_1}{\partial y^2} = 0, \quad x > 0, \quad y > 0,$$
$$V_1(0,y) = g(y), \quad y > 0,$$
$$\frac{\partial V_1(x,0)}{\partial y} = 0, \quad x > 0,$$

$$\frac{\partial^2 V_2}{\partial x^2} + \frac{\partial^2 V_2}{\partial y^2} = 0, \quad x > 0, \quad y > 0,$$
$$V_2(0,y) = 0, \quad y > 0,$$
$$\frac{\partial V_2(x,0)}{\partial y} = f(x), \quad x > 0.$$

To find $V_1(x,y)$ we apply Fourier cosine transform 11.40a (with respect to y) to its PDE and use property 11.43b,

$$\frac{d^2\tilde{V}_1}{dx^2} - \omega^2 \tilde{V}_1(x,\omega) = 0, \quad x > 0.$$

This transform function $\tilde{V}_1(x,\omega)$ is also subject to

$$\tilde{V}_1(0,\omega) = \tilde{g}(\omega).$$

A general solution of the ODE is

$$\tilde{V}_1(x,\omega) = Ae^{\omega x} + Be^{-\omega x}.$$

For $\tilde{V}_1(x,\omega)$ to remain bounded as $x \to \infty$, A must be zero, and the boundary condition then implies that $B = \tilde{g}(\omega)$. Hence,

$$\tilde{V}_1(x,\omega) = \tilde{g}(\omega)e^{-\omega x}.$$

To invert this transform, first recall from to Example 11.17 that

$$\mathcal{F}_c\left\{e^{-ay}\right\}(\omega) = \frac{a}{a^2+\omega^2} \qquad \text{when } a > 0.$$

With Exercise 19 in Section 11.5, we can say that

$$\mathcal{F}_c^{-1}\{e^{-a\omega}\}(y) = \frac{2}{\pi}\frac{a}{a^2+y^2}.$$

Convolution property 11.44c, now gives

$$V_1(x,y) = \frac{1}{2}\int_0^\infty g(v)\left(\frac{2}{\pi}\right)\left[\frac{x}{(y-v)^2+x^2} + \frac{x}{(y+v)^2+x^2}\right] dv$$
$$= \frac{x}{\pi}\int_0^\infty g(v)\left[\frac{1}{x^2+(y-v)^2} + \frac{1}{x^2+(y+v)^2}\right] dv.$$

Taking Fourier sine transforms with respect to x in order to find $V_1(x,y)$ leads to a nonhomogeneous ODE in $\tilde{V}_1(\omega,y)$ that is more difficult to solve.

To find $V_2(x,y)$ we apply the Fourier sine transform with respect to x to its PDE and use property 11.43d,

$$\frac{d^2\tilde{V}_2}{dy^2} - \omega^2\tilde{V}_2(\omega,y) = 0.$$

The transform must also satisfy

$$\frac{d\tilde{V}_2(\omega,0)}{dy} = \tilde{f}(\omega).$$

A general solution of the ODE is

$$\tilde{V}_2(\omega,y) = Ae^{\omega y} + Be^{-\omega y}.$$

For $\tilde{V}_2(\omega,y)$ to remain bounded as $y \to \infty$, A must be zero, and the boundary condition on $\tilde{V}_2$ then implies that $B = -\tilde{f}(\omega)/\omega$. Hence,

$$\tilde{V}_2(\omega,y) = -\frac{\tilde{f}(\omega)}{\omega}e^{-\omega y}$$

and

$$V_2(x,y) = \frac{2}{\pi}\int_0^\infty -\frac{\tilde{f}(\omega)}{\omega}e^{-\omega y}\sin\omega x\,d\omega.$$

The final solution is

$$V(x,y) = \frac{x}{\pi}\int_0^\infty g(v)\left[\frac{1}{x^2+(y-v)^2} + \frac{1}{x^2+(y+v)^2}\right]dv + \frac{2}{\pi}\int_0^\infty -\frac{\tilde{f}(\omega)}{\omega}e^{-\omega y}\sin\omega x\,d\omega. \bullet$$

Time-dependent heat and vibration problems on infinite or semi-infinite intervals require Fourier transforms. The boundary value problem in Example 11.22 also requires Fourier transforms since both x and y are on semi-infinite intervals. When solving Laplace's (or Poisson's) equation in the xy-plane where one of x or y is of finite extent, it may not be advantageous to introduce Fourier transforms; separation of variables or finite Fourier transforms may be preferable. We illustrate in the following example.

Example 11.23 A thin plate has edges along $y = 0$, $y = L'$, and $x = 0$ for $0 \le y \le L'$. The other edge is so far to the right that its effect may be considered negligible. Assuming no heat flow in the z-direction, find the steady-state temperature inside the plate (for $x > 0$, $0 < y < L'$) if sides $y = 0$ and $y = L'$ are held at constant temperature U_0°C, and side $x = 0$ has temperature $f(y)$, $0 \le y \le L'$.

Solution The boundary value problem for steady-state temperature $U(x,y)$ is

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = 0, \quad x > 0, \quad 0 < y < L', \tag{11.60a}$$

$$U(x,0) = U(x,L') = U_0, \quad x > 0, \tag{11.60b}$$

$$U(0,y) = f(y), \quad 0 < y < L'. \tag{11.60c}$$

The finite Fourier transform associated with y is

$$\widetilde{f}(\lambda_n) = \int_0^{L'} f(y)Y_n(y)\,dy,$$

where $\lambda_n^2 = n^2\pi^2/L'^2$ and $Y_n(y) = \sqrt{2/L'}\sin\lambda_n y$ are eigenpairs of the Sturm-Liouville system

$$Y'' + \lambda^2 Y = 0, \quad 0 < y < L', \qquad Y(0) = Y(L') = 0.$$

When we apply the transform to the PDE, and use integration by parts,

$$\begin{aligned}\frac{\partial^2\widetilde{U}}{\partial x^2} &= -\int_0^{L'} \frac{\partial^2 U}{\partial y^2}Y_n(y)\,dy = -\left\{\frac{\partial U}{\partial y}Y_n\right\}_0^{L'} + \int_0^{L'} \frac{\partial U}{\partial y}Y_n'\,dy \\ &= \left\{UY_n'\right\}_0^{L'} - \int_0^{L'} UY_n''\,dy = U_0Y_n'(L') - U_0Y_n'(0) - \int_0^{L'} U\left(-\lambda_n^2 Y_n\right)\,dy \\ &= U_0[Y_n'(L') - Y_n'(0)] + \lambda_n^2\widetilde{U}.\end{aligned}$$

Thus, $\widetilde{U}(x,\lambda_n)$ must satisfy the ODE

$$\frac{d^2\widetilde{U}}{dx^2} - \lambda_n^2\widetilde{U} = U_0[Y_n'(L') - Y_n'(0)],$$

subject to

$$\widetilde{U}(0,\lambda_n) = \widetilde{f}(\lambda_n).$$

A general solution of the differential equation is

$$\widetilde{U}(x,\lambda_n) = Ae^{\lambda_n x} + Be^{-\lambda_n x} - \frac{U_0[Y_n'(L') - Y_n'(0)]}{\lambda_n^2}.$$

For this to remain bounded as $x \to \infty$, we must set $A = 0$, in which case the boundary condition requires

$$\widetilde{f}(\lambda_n) = B - \frac{U_0[Y_n'(L') - Y_n'(0)]}{\lambda_n^2}.$$

Thus,

$$\widetilde{U}(x,\lambda_n) = \widetilde{f}(\lambda_n)e^{-\lambda_n x} - \frac{U_0[Y_n'(L') - Y_n'(0)]}{\lambda_n^2}(1 - e^{-\lambda_n x}).$$

The inverse finite Fourier transform now gives

$$\begin{aligned}U(x,y) &= \sum_{n=1}^{\infty}\widetilde{U}(x,\lambda_n)Y_n(y) \\ &= \sum_{n=1}^{\infty}\left\{\widetilde{f}(\lambda_n)e^{-\lambda_n x} - \frac{U_0[Y_n'(L') - Y_n'(0)]}{\lambda_n^2}(1 - e^{-\lambda_n x})\right\}\sqrt{\frac{2}{L'}}\sin\lambda_n y \\ &= \sqrt{\frac{2}{L'}}\sum_{n=1}^{\infty}\widetilde{f}(\lambda_n)e^{-n\pi x/L'}\sin\frac{n\pi y}{L'} + \frac{2U_0}{\pi}\sum_{n=1}^{\infty}\frac{[1 + (-1)^{n+1}]}{n}(1 - e^{-n\pi x/L'})\sin\frac{n\pi y}{L'}.\end{aligned}$$

Since

$$\widetilde{1} = \int_0^{L'}\sqrt{\frac{2}{L'}}\sin\frac{n\pi y}{L'}dy = \frac{\sqrt{2L'}[1 + (-1)^{n+1}]}{n\pi},$$

it follows that

$$U(x,y) = U_0 + \sqrt{\frac{2}{L'}} \sum_{n=1}^{\infty} \left\{ \widetilde{f}(\lambda_n) - \frac{\sqrt{2L'}U_0[1+(-1)^{n+1}]}{n\pi} \right\} e^{-n\pi x/L'} \sin \frac{n\pi y}{L'}. \bullet$$

EXERCISES 11.6

Part A Heat Conduction

1. Use a Fourier transform to find an integral representation for the solution of the heat conduction problem

$$\begin{aligned} \frac{\partial U}{\partial t} &= k\frac{\partial^2 U}{\partial x^2}, \quad x > 0, \quad t > 0, \\ U(0,t) &= \overline{U} = \text{constant}, \quad t > 0, \\ U(x,0) &= 0, \quad x > 0. \end{aligned}$$

(Hint: See Exercise 20 in Section 11.5 when inverting the transform.) Is the solution the same as that in Example 10.9?

(b) Plot the solution on the interval $0 \le x \le 5$ with $k = 10^{-6}$ and $\overline{U} = 1$ for $t = 10^5$ and $t = 10^6$.

(c) Comment on the possibility of using the transformation $W = U - \overline{U}$ to remove the nonhomogeneity from the boundary condition.

2. (a) Use a Fourier transform to find an integral representation for the solution of the heat conduction problem

$$\begin{aligned} \frac{\partial U}{\partial t} &= k\frac{\partial^2 U}{\partial x^2}, \quad x > 0, \quad t > 0, \\ U_x(0,t) &= -\kappa^{-1}Q_0 = \text{constant}, \quad t > 0, \\ U(x,0) &= 0, \quad x > 0. \end{aligned}$$

(Hint: See Exercise 20 in Section 11.5 when inverting the transform.) Plot the solution on the interval $0 \le x \le 5$ with $k = 10^{-6}$, $\kappa = 10$, and $Q_0 = 1000$ for $t = 10^5$ and $t = 10^6$.

(b) Describe the temperature of the left end of the rod.

3. (a) Use a Fourier transform to find an integral representation for the solution of the heat conduction problem

$$\begin{aligned} \frac{\partial U}{\partial t} &= k\frac{\partial^2 U}{\partial x^2} + \frac{k}{\kappa}g(x,t), \quad x > 0, \quad t > 0, \\ U(0,t) &= f_1(t), \quad t > 0, \\ U(x,0) &= f(x), \quad x > 0. \end{aligned}$$

(b) Simplify the solution in part (a) when $g(x,t) \equiv 0$, $f_1(t) \equiv 0$, and $f(x) = U_0 =$ constant.

(c) Simplify the solution in part (a) when $g(x,t) \equiv 0$, $f(x) \equiv 0$, and $f_1(t) = \overline{U} =$ constant. Is it the solution of Exercise 1?

4. (a) Use a Fourier transform to find an integral representation for the solution of the heat conduction problem

$$\begin{aligned} \frac{\partial U}{\partial t} &= k\frac{\partial^2 U}{\partial x^2} + \frac{k}{\kappa}g(x,t), \quad x>0, \quad t>0,\\ U_x(0,t) &= -\kappa^{-1}f_1(t), \quad t>0,\\ U(x,0) &= f(x), \quad x>0. \end{aligned}$$

(b) Simplify the solution in part (a) when $g(x,t) \equiv 0$, $f_1(t) \equiv 0$, and $f(x) = U_0 =$ constant.
(c) Simplify the solution in part (a) when $g(x,t) \equiv 0$, $f(x) \equiv 0$, and $f_1(t) = Q_0 =$ constant. Is it the solution of Exercise 2?

5. Use the Fourier transform of Exercise 22 in Section 11.5 to find an integral representation for the solution of the heat conduction problem

$$\begin{aligned} \frac{\partial U}{\partial t} &= k\frac{\partial^2 U}{\partial x^2}, \quad x>0, \quad t>0,\\ -\kappa\frac{\partial U(0,t)}{\partial x} + \mu U(0,t) &= \mu U_m = \text{constant}, \quad t>0,\\ U(x,0) &= 0, \quad x>0. \end{aligned}$$

6. Use the Fourier transform of Exercise 22 in Section 11.5 to find an integral representation for the solution of the heat conduction problem

$$\begin{aligned} \frac{\partial U}{\partial t} &= k\frac{\partial^2 U}{\partial x^2} + \frac{k}{\kappa}g(x,t), \quad x>0, \quad t>0,\\ -\kappa\frac{\partial U(0,t)}{\partial x} + \mu U(0,t) &= \mu f_1(t), \quad t>0,\\ U(x,0) &= f(x), \quad x>0. \end{aligned}$$

Part B Vibrations

7. Solve the vibration problem of Example 11.20 if a unit force acts at the point $x = x_0$ on the string for all $t > 0$.

8. Repeat Example 11.20 if the Dirichlet boundary condition at $x = 0$ is replaced by the Neumann condition

$$y_x(0,t) = -\tau^{-1}f_1(t).$$

Constant τ is the tension in the string. This boundary condition describes the situation where the end $x = 0$ of the string, taken as massless, moves vertically with tension and an external force $f_1(t)$ acting on the end.

Part C Potential, Steady-state Heat Conduction, Static Deflection of Membranes

9. Solve the boundary value problem

$$\begin{aligned} \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} &= 0, \quad x>0, \quad 0<y<L,\\ V(0,y) &= 0, \quad 0<y<L,\\ V(x,0) &= f(x), \quad x>0,\\ V(x,L) &= g(x), \quad x>0. \end{aligned}$$

10. Solve the boundary value problem in Exercise 9 if the boundary condition along the x-axis is Neumann $V_y(x,0) = f(x)$.

11. Solve the boundary value problem in Exercise 9 if the boundary condition along $y = L$ is Neumann $V_y(x,L) = g(x)$.

12. Solve the boundary value problem in Exercise 9 if the boundary condition along $x = 0$ is homogeneous Neumann $V_x(0,y) = 0$.

13. Solve the boundary value problem

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0, \quad x > 0, \quad y > 0,$$
$$V(0,y) = g(y), \quad y > 0,$$
$$V(x,0) = f(x), \quad x > 0.$$

14. Solve the boundary value problem for potential in the semi-infinite strip $0 < y < L$, $x > 0$ when potential on $y = 0$ and $y = L$ is zero and that on $x = 0$ is $f(y)$. Simplify the solution when $f(y)$ is constant.

15. (a) Use Exercises 9 and 14 to solve Exercise 14 when potentials on $x = 0$, $y = 0$, and $y = L'$ are $f(y)$, $g_1(x)$, and $g_2(x)$, respectively.
(b) Try to solve the problem using a Fourier sine transform on x.
(c) Try to solve the problem using a finite Fourier transform on y.

16. A thin plate has edges along $y = 0$, $y = L$, and $x = 0$ for $0 \le y \le L$. The other edge is so far to the right that its effect may be considered negligible. Assuming no heat flow in the z-direction, find the steady-state temperature inside the plate (for $x > 0$, $0 < y < L$) if side $y = 0$ is held at temperature 0°C, side $y = L$ is insulated, and, along $x = 0$:
(a) temperature is held at a constant U_0°C.
(b) heat is added to the plate at a constant rate $Q_0 > 0$ W/m^2 over the interval $0 < y < L/2$ and extracted at the same rate for $L/2 < y < L$.
(c) heat is transferred to a medium at constant temperature U_m according Newton's law of cooling.

17. What are the solutions to Exercise 16 if edge $y = 0$ is insulated instead of held at temperature 0°C.

18. Does the function

$$U(x,y) = \begin{cases} -Q_0 x/\kappa, & 0 < y < L/2 \\ Q_0 x/\kappa, & L/2 < y < L \end{cases}$$

satisfy the PDE and the boundary conditions on $x = 0$, $y = 0$, and $y = L$ in Exercise 16(b)? Why is this not the solution?

19. (a) A thin plate has edges along $y = 0$, $y = L$, and $x = 0$ for $0 \le y \le L$. The other edge is so far to the right that its effect may be considered negligible. Assuming no heat flow in the z-direction, find the steady-state temperature inside the plate (for $x > 0$, $0 < y < L$) if side $x = 0$ is held at temperature $f(y)$, side $y = 0$ is held at temperature zero, and along side $y = L$ heat is transferred according to Newton's law of cooling to a medium at constant temperature U_m.
(b) Simplify the solution in part (a) when $U_m = 0$ and $f(y) = U_0$, a constant.

20. Repeat Exercise 19 when side $y = 0$ is insulated.

21. (a) A uniform charge distribution of density σ coulombs per cubic metre occupies the region bounded by the planes $x = 0$, $y = 0$, and $x = L$ $(y \geq 0)$. If the planes $x = 0$ and $y = 0$ are kept at zero potential and $x = L$ is maintained at a constant potential V_L, find the potential between the planes using:

(i) a finite Fourier transform.

(ii) a transformation to remove the constant nonhomogeneities σ and V_L.

(b) Can we apply a Fourier sine transform with respect to y?

22. If the charge distribution in Exercise 21 is a function of y, $\sigma(y) = e^{-y}$, find the potential between the plates.

23. Solve Exercise 22 when $V_L = 0$, using

(a) a finite Fourier transform (b) the Fourier sine transform