§5.2 Generalized Fourier Series

In Chapters 3 and 4 we learned how to express functions $f(x)$, which are piecewise smooth on the interval $0 \le x \le L$, in the form of Fourier sine series

$$\frac{f(x+) + f(x-)}{2} = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L} \quad \text{where} \quad b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx. \tag{5.8}$$

We regard the Fourier coefficients b_n as the components of the function $f(x)$ with respect to the basis functions $\{\sin(n\pi x/L)\}$. In Section 5.1 we discovered that the $\sin(n\pi x/L)$ are eigenfunctions of Sturm-Liouville system 5.1, and it has become our practice to replace eigenfunctions with normalized eigenfunctions, namely $\sqrt{2/L}\sin(n\pi x/L)$. Representation 5.8 can easily be replaced by an equivalent expression in terms of these normalized eigenfunctions,

$$\frac{f(x+) + f(x-)}{2} = \sum_{n=1}^{\infty} c_n \left(\sqrt{\frac{2}{L}}\sin\frac{n\pi x}{L}\right) \quad \text{where} \quad c_n = \int_0^L f(x)\left(\sqrt{\frac{2}{L}}\sin\frac{n\pi x}{L}\right)dx. \tag{5.9}$$

Constants c_n are the components of $f(x)$ with respect to the orthonormal basis $\{\sqrt{2/L}\sin(n\pi x/L)\}$. Equation 5.8 should be compared with equation 3.3 in Section 3.1, together with the fact that the length of $\sin(n\pi x/L)$ is $\sqrt{L/2}$. Equation 5.9 is analogous to equation 3.1.

The same function $f(x)$ can be represented by a Fourier cosine series in terms of normalized eigenfunctions of system 5.2,

$$\frac{f(x+) + f(x-)}{2} = \frac{c_0}{\sqrt{L}} + \sum_{n=1}^{\infty} c_n \left(\sqrt{\frac{2}{L}}\cos\frac{n\pi x}{L}\right) \tag{5.10a}$$

where

$$c_0 = \int_0^L f(x)\left(\frac{1}{\sqrt{L}}\right)dx \quad \text{and} \quad c_n = \int_0^L f(x)\left(\sqrt{\frac{2}{L}}\cos\frac{n\pi x}{L}\right)dx, \quad n > 0. \tag{5.10b}$$

A natural question to ask now is the following: Given a function $f(x)$, defined on the interval $a \le x \le b$, and given a Sturm-Liouville system on the same interval, is it always possible to express $f(x)$ in terms of the orthonormal eigenfunctions of the Sturm-Liouville system? It is still not clear that every Sturm-Liouville system has an infinity of eigenfunctions, but, as we shall see, this is indeed the case. We wish then to investigate the possibility of finding coefficients c_n such that on $a \le x \le b$,

$$f(x) = \sum_{n=1}^{\infty} c_n y_n(x), \tag{5.11}$$

where $y_n(x)$ are the orthonormal eigenfunctions of Sturm-Liouville system 5.3. If we formally multiply equation 5.11 by $p(x)y_m(x)$, and integrate term-by-term between $x = a$ and $x = b$,

$$\int_a^b p(x)f(x)y_m(x)\,dx = \sum_{n=1}^{\infty} c_n \int_a^b p(x)y_n(x)y_m(x)\,dx.$$

Because of the orthogonality of eigenfunctions, only the m^{th} term in the series does not vanish, and therefore

$$\int_a^b p(x)f(x)y_m(x)\,dx = c_m. \tag{5.12}$$

This has been strictly a formal procedure. It has illustrated that *if* $f(x)$ can be represented in form 5.11, and *if* the series is suitably convergent, coefficients c_n can be calculated according to formula 5.12. What we must answer is the converse question: If coefficients c_n are calculated according to 5.12, where $y_n(x)$ are orthonormal eigenfunctions of a Sturm-Liouville system, does series 5.11 converge to $f(x)$? This question is answered in the following theorem.

Theorem 5.2 Let p, q, r, r', and $(pr)''$ be real and continuous functions of x for $a \le x \le b$, and let $p > 0$ and $r > 0$ for $a \le x \le b$. Let l_1, l_2, h_1, and h_2 be real constants independent of λ. Then Sturm-Liouville system 5.3 has a countable infinity of simple eigenvalues $\lambda_1 < \lambda_2 < \lambda_3 < \cdots$ (all real), not more than a finite number of which are negative, and $\lim_{n\to\infty} \lambda_n = \infty$. Corresponding orthonormal eigenfunctions $y_n(x)$ are such that $y_n(x)$ and $y_n{}'(x)$ are continuous and $|y_n(x)|$ and $|\lambda_n{}^{-1/2} y_n{}'(x)|$ are uniformly bounded with respect to x and n. If $f(x)$ is piecewise smooth on $a \le x \le b$, then for any x in $a < x < b$,

$$\frac{f(x+) + f(x-)}{2} = \sum_{n=1}^{\infty} c_n y_n(x), \qquad \text{where} \qquad c_n = \int_a^b p(x)f(x)y_n(x)\,dx. \tag{5.13}$$

Series 5.13 is called the **generalized Fourier series** for $f(x)$ with respect to the eigenfunctions $y_n(x)$, and the c_n are the **generalized Fourier coefficients**. They are the components of $f(x)$ with respect to the orthonormal basis of eigenfunctions $\{y_n(x)\}$. Notice the similarity between this theorem and Theorem 3.2 in Section 3.1 for Fourier series. Both guarantee pointwise convergence of Fourier series for a piecewise smooth function to the value of the function at a point of continuity of the function, and to the average value of right- and left-hand limits at a point of discontinuity. Because the eigenfunctions in Theorem 3.2 of Section 3.1 are periodic, convergence is also assured at the end points of the interval $0 \le x \le 2L$. This is not the case in Theorem 5.2 above. Eigenfunctions are not generally periodic, and convergence at $x = a$ and $x = b$ is not guaranteed. It should be clear, however, that when $l_1 = 0$ (in which case $y_n(a) = 0$) convergence of the series in 5.13 at $x = a$ can be expected only if $f(a) = 0$ also. A similar statement can be made at $x = b$.

Because series 5.13 is a representation of the function $f(x)$ in terms of normalized eigenfunctions of a regular Sturm-Liouville system, it is also called an **eigenfunction expansion** of $f(x)$. We use both terms, namely, generalized Fourier series and eigenfunction expansion, freely and interchangeably.

We say that the normalized eigenfunctions of Sturm-Liouville system 5.3 form a **complete set** for the space of piecewise smooth functions on the interval $a \le x \le b$; this means that every piecewise smooth function can be expressed in a convergent series of the eigenfunctions.

When a regular Sturm-Liouville system satisfies the conditions of this theorem as well as the conditions that $q(x) \ge 0$ for $a \le x \le b$, and $l_1 h_1 \ge 0$, $l_2 h_2 \ge 0$, it is said to be a **proper Sturm-Liouville system**. For such a system we shall take l_1, l_2, h_1, and h_2 all nonnegative, in which case we can prove the following corollary.

Corollary All eigenvalues of a proper Sturm-Liouville system are nonnegative. Furthermore, zero is an eigenvalue of a proper Sturm-Liouville system only when $q(x) \equiv 0$ and $h_1 = h_2 = 0$.

Proof Let λ and $y(x)$ be an eigenpair of a regular Sturm-Liouville system. Multiplication of differential equation 5.3a by $y(x)$ and integration from $x = a$ to $x = b$ gives

$$\begin{aligned}\lambda \int_a^b p(x)y^2(x)\,dx &= \int_a^b q(x)y^2(x)\,dx - \int_a^b y(x)[r(x)y'(x)]'\,dx \\ &= \int_a^b q(x)y^2(x)\,dx - \Big\{r(x)y(x)y'(x)\Big\}_a^b + \int_a^b r(x)[y'(x)]^2\,dx.\end{aligned}$$

When we solve boundary conditions 5.3b,c for $y'(b)$ and $y'(a)$ and substitute into the second term on the right, we obtain

$$\begin{aligned}\lambda \int_a^b p(x)y^2(x)\,dx = \int_a^b q(x)y^2(x)\,dx &+ \int_a^b r(x)[y'(x)]^2\,dx \\ &+ \frac{h_2}{l_2}r(b)y^2(b) + \frac{h_1}{l_1}r(a)y^2(a).\end{aligned}$$

When the Sturm-Liouville system is proper, every term on the right is nonnegative, as is the integral on the left, and therefore $\lambda \geq 0$. (If either $l_1 = 0$ or $l_2 = 0$, the corresponding terms in the above equation are absent and the result is the same.)

Furthermore, if $\lambda = 0$ is an eigenvalue, then each of the four terms on the right side of the above equation must vanish separately. The first requires that $q(x) \equiv 0$ and the second that $y'(x) = 0$. But the fact that $y(x)$ is constant implies that the last two terms can vanish only if $h_1 = h_2 = 0$. ∎

Since eigenvalues of a proper Sturm-Liouville system must be nonnegative, we may replace λ by λ^2 in differential equation 5.3a whenever it is convenient to do so,

$$\frac{d}{dx}\left[r(x)\frac{dy}{dx}\right] + [\lambda^2 p(x) - q(x)]y = 0, \quad a < x < b.$$

This often has the advantage of eliminating square roots in calculations.

Example 5.4 Expand the function $f(x) = L - x$ in terms of normalized eigenfunctions of the Sturm-Liouville system of Example 5.1.

Solution According to Example 5.1, eigenfunctions of the Sturm-Liouville system are $\sin\dfrac{(2n-1)\pi x}{2L}$. Because

$$\left\|\sin\frac{(2n-1)\pi x}{2L}\right\|^2 = \int_0^L \left[\sin\frac{(2n-1)\pi x}{2L}\right]^2 dx = \frac{L}{2},$$

normalized eigenfunctions are $X_n(x) = \sqrt{\dfrac{2}{L}}\sin\dfrac{(2n-1)\pi x}{2L}$. In terms of these eigenfunctions, the generalized Fourier series for $f(x) = L - x$ is

$$L - x = \sum_{n=1}^{\infty} c_n X_n(x),$$

where

$$c_n = \int_0^L (L-x)\sqrt{\frac{2}{L}} \sin \frac{(2n-1)\pi x}{2L} dx = \frac{2\sqrt{2}L^{3/2}}{\pi^2}\left[\frac{\pi}{2n-1} + \frac{2(-1)^n}{(2n-1)^2}\right].$$

Thus,

$$L - x = \frac{2\sqrt{2}L^{3/2}}{\pi^2} \sum_{n=1}^{\infty} \left[\frac{\pi}{2n-1} + \frac{2(-1)^n}{(2n-1)^2}\right] \sqrt{\frac{2}{L}} \sin \frac{(2n-1)\pi x}{2L}.$$

Theorem 5.2 guarantees convergence of the series to $L-x$ for $0 < x < L$. It obviously does not converge to $L-x$ at $x=0$, but it does converge to $L-x$ at $x=L$. This follows from the facts that

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n-1} = \frac{\pi}{4} \qquad \text{and} \qquad \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}.$$

Figure 5.1 shows a few partial sums of this series to illustrate convergence of the series to $L-x$. It is very slow because of the term $\pi/(2n-1)$.•

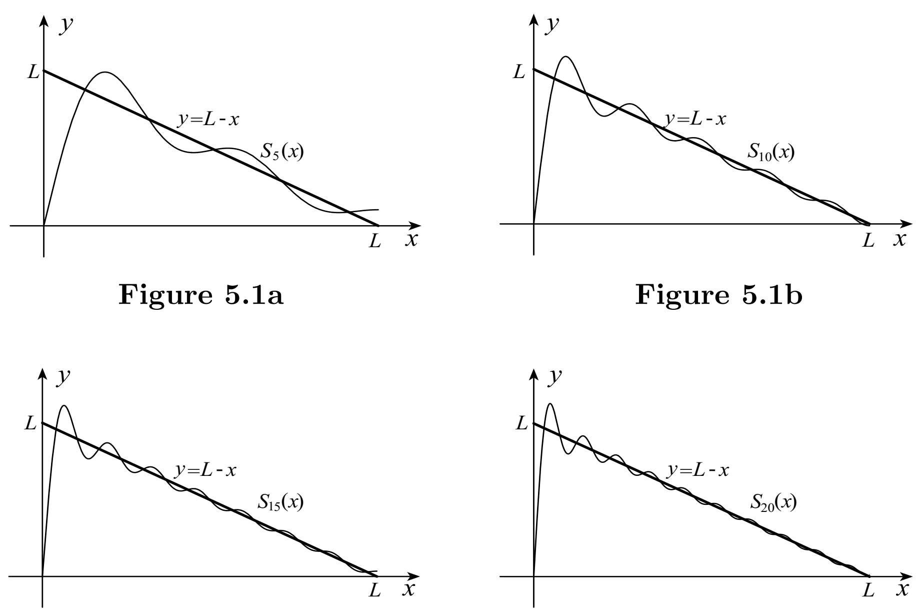

Figure 5.1a **Figure 5.1b**

Figure 5.1c **Figure 5.1d**

In the examples of Chapter 4, when separation of variables was applied to (initial) boundary value problems, all boundary conditions in a given problem were either of Dirichlet type or Neumann type. These led to Fourier sine and cosine series, series that we now know are eigenfunction expansions in terms of eigenfunctions of Sturm-Liouville systems 5.1 and 5.2. We did not consider problems with Robin conditions, but in some of the exercises, we mixed Dirichlet and Neumann conditions. We were able to do so because of the results in Exercises 20 and 21 of Section 3.2. With our results on Sturm-Liouville systems in this section, we will be well prepared to tackle any combination of Dirichlet, Neumann, and Robin boundary conditions.

A proper Sturm-Liouville system that arises repeatedly in our discussions is

$$\frac{d^2X}{dx^2} + \lambda^2 X = 0, \quad 0 < x < L, \tag{5.14a}$$

$$-l_1 X'(0) + h_1 X(0) = 0, \tag{5.14b}$$

$$l_2 X'(L) + h_2 X(L) = 0. \tag{5.14c}$$

where l_1, l_2, h_1, and h_2 are non-negative constants. (Systems 5.1 and 5.2 are special cases of 5.14 when $l_1 = l_2 = 0$ and $h_1 = h_2 = 0$, respectively. Examples 5.1 and 5.4 contain the special case of $l_1 = h_2 = 0$ and $l_2 = h_1 = 1$.) We consider here the most general case, in which $h_1 h_2 l_1 l_2 \neq 0$; special cases in which one or two of h_1, h_2, l_1, and l_2 vanish are tabulated later. In the general case when $h_1 h_2 l_1 l_2 \neq 0$, we could divide boundary condition 5.14b by either l_1 or h_1. This would lead to a boundary condition with only one arbitrary constant (h_1/l_1 or l_1/h_1). Likewise, we could divide boundary condition 5.14c by l_2 or h_2 and express the boundary condition in terms of the ratio h_2/l_2 or l_2/h_2. However, when this is done, it is not quite so transparent how to specialize the results in the cases in which one or two of h_1, h_2, l_1, and l_2 vanish. For this reason, we prefer to leave 5.14b,c in their present forms.

We are justified in representing the eigenvalues of system 5.14 by λ^2 rather than λ, because all eigenvalues of a proper Sturm-Liouville system are nonnegative. A general solution of differential equation 5.14a is

$$X(x) = A\cos\lambda x + B\sin\lambda x, \tag{5.15}$$

and when we impose boundary conditions 5.14b,c,

$$-l_1\lambda B + h_1 A = 0, \tag{5.16a}$$

$$l_2(-A\lambda\sin\lambda L + B\lambda\cos\lambda L) + h_2(A\cos\lambda L + B\sin\lambda L) = 0. \tag{5.16b}$$

We solve equation 5.16a for $B = h_1 A/(l_1\lambda)$ and substitute into 5.16b. After rearrangement, we obtain

$$\tan\lambda L = \frac{\lambda\left(\dfrac{h_1}{l_1} + \dfrac{h_2}{l_2}\right)}{\lambda^2 - \dfrac{h_1 h_2}{l_1 l_2}}, \tag{5.17}$$

the equation that must be satisfied by λ. We denote by λ_n $(n = 1, 2, \ldots)$ eigenvalues of this transcendental equation, although, in fact, λ_n^2 are the eigenvalues of the Sturm-Liouville system. Corresponding to these eigenvalues are orthonormal eigenfunctions

$$X_n(x) = \frac{1}{N}\left(\cos\lambda_n x + \frac{h_1}{\lambda_n l_1}\sin\lambda_n x\right), \tag{5.18a}$$

where

$$N^2 = \int_0^L \left(\cos\lambda_n x + \frac{h_1}{\lambda_n l_1}\sin\lambda_n x\right)^2 dx. \tag{5.18b}$$

In Exercise 1, integration is shown to lead to

$$2N^2 = \left[1 + \left(\frac{h_1}{\lambda_n l_1}\right)^2\right]\left[L + \frac{h_2/l_2}{\lambda_n^2 + (h_2/l_2)^2}\right] + \frac{h_1/l_1}{\lambda_n^2}. \tag{5.18c}$$

Of the nine possible combinations of boundary conditions at $x = 0$ and $x = L$, we have considered only one, the most general in which none of h_1, h_2, l_1, and l_2 vanishes. Results for the remaining eight cases can be obtained from equations 5.17 and 5.18, or by similar analyses; they are tabulated in Table 5.1.

Each eigenvalue equation in Table 5.1 is unchanged if λ is replaced by $-\lambda$, so that for every positive solution λ of the equation, $-\lambda$ is also a solution. Since NX_n is invariant (up to a sign change) by the substitution of $-\lambda_n$ for λ_n, it is necessary only to consider nonnegative solutions of the eigenvalue equations. This agrees with the fact that eigenvalues of the Sturm-Liouville system are λ_n^2 and that there cannot be two linearly independent eigenfunctions corresponding to the same eigenvalue. Table 5.1 gives the eigenvalues explicitly in only four of the nine cases. Eigenvalues in the remaining five cases are illustrated geometrically below.

If $h_1h_2l_1l_2 \neq 0$, eigenvalues are illustrated graphically in Figure 5.2 as points of intersection of the curves

$$y = \tan\lambda L, \qquad y = \frac{\lambda(h_1/l_1 + h_2/l_2)}{\lambda^2 - h_1h_2/(l_1l_2)}.$$

It might appear that $\lambda = 0$ is an eigenvalue in this case. However, the corollary to Theorem 5.2 indicates that zero is an eigenvalue only when $h_1 = h_2 = 0$. This can also be verified using conditions 5.16, which led to the eigenvalue equation (see Exercise 3).

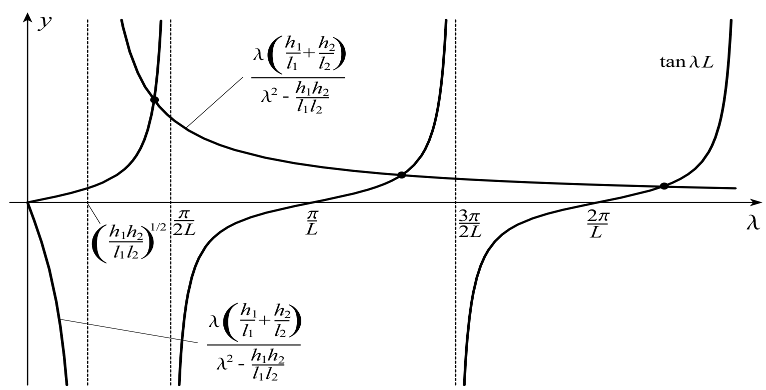

Figure 5.2

Condition at $x=0$	Condition at $x=L$	Eigenvalue Equation	NX_n	$2N^2$
$h_1l_1 \neq 0$	$h_2l_2 \neq 0$	$\tan\lambda L = \dfrac{\lambda\left(\dfrac{h_1}{l_1}+\dfrac{h_2}{l_2}\right)}{\lambda^2-\dfrac{h_1h_2}{l_1l_2}}$	$\cos\lambda_n x + \dfrac{h_1}{\lambda_n l_1}\sin\lambda_n x$	$\dfrac{h_1/l_1}{\lambda_n^2}+\left[1+\left(\dfrac{h_1}{\lambda_n l_1}\right)^2\right] \times\left[L+\dfrac{h_2/l_2}{\lambda_n^2+(h_2/l_2)^2}\right]$
$h_1l_1 \neq 0$	$h_2=0$ $(l_2=1)$	$\tan\lambda L = \dfrac{h_1}{\lambda l_1}$	$\dfrac{\cos\lambda_n(L-x)}{\cos\lambda_n L}$	$L\left[1+\left(\dfrac{h_1}{\lambda_n l_1}\right)^2\right]+\dfrac{h_1/l_1}{\lambda_n^2}$
$h_1l_1 \neq 0$	$l_2=0$ $(h_2=1)$	$\cot\lambda L = -\dfrac{h_1}{\lambda l_1}$	$\dfrac{\sin\lambda_n(L-x)}{\sin\lambda_n L}$	$L\left[1+\left(\dfrac{h_1}{\lambda_n l_1}\right)^2\right]+\dfrac{h_1/l_1}{\lambda_n^2}$
$h_1=0$ $(l_1=1)$	$h_2l_2 \neq 0$	$\tan\lambda L = \dfrac{h_2}{\lambda l_2}$	$\cos\lambda_n x$	$L+\dfrac{h_2/l_2}{\lambda_n^2+(h_2/l_2)^2}$
$h_1=0$ $(l_1=1)$	$h_2=0$ $(l_2=1)$	$\sin\lambda L=0$ $\lambda_n=\dfrac{n\pi}{L},\ n=0,1,2,\ldots$	$\cos\lambda_n x$	$L\ (n\neq 0)$ $2L\ (n=0)$
$h_1=0$ $(l_1=1)$	$l_2=0$ $(h_2=1)$	$\cos\lambda L=0$ $\lambda_n=\dfrac{(2n-1)\pi}{2L},\ n=1,2,\ldots$	$\cos\lambda_n x$	L
$l_1=0$ $(h_1=1)$	$h_2l_2 \neq 0$	$\cot\lambda L = -\dfrac{h_2}{\lambda l_2}$	$\sin\lambda_n x$	$L+\dfrac{h_2/l_2}{\lambda_n^2+(h_2/l_2)^2}$
$l_1=0$ $(h_1=1)$	$h_2=0$ $(l_2=1)$	$\cos\lambda L=0$ $\lambda_n=\dfrac{(2n-1)\pi}{2L},\ n=1,2,\ldots$	$\sin\lambda_n x$	L
$l_1=0$ $(h_1=1)$	$l_2=0$ $(h_2=1)$	$\sin\lambda L=0$ $\lambda_n=\dfrac{n\pi}{L},\ n=1,2,\ldots$	$\sin\lambda_n x$	L

Table 5.1

If $h_1l_1 \neq 0$ and $h_2 = 0$ (in which case we set $l_2 = 1$), eigenvalues are illustrated graphically in Figure 5.3 as points of intersection of the curves

$$y = \tan\lambda L, \qquad y = h_1/(\lambda l_1).$$

A similar situation arises when $h_2l_2 \neq 0$ and $h_1 = 0$.

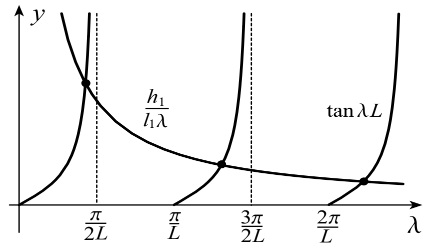

Figure 5.3

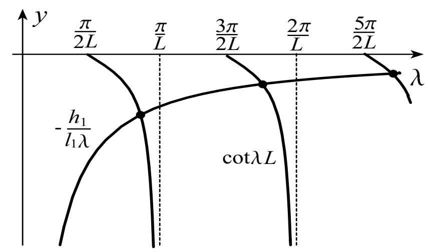

Figure 5.4

If $h_1 l_1 \neq 0$ and $l_2 = 0$ (in which case we set $h_2 = 1$), eigenvalues are illustrated graphically in Figure 5.4 as points of intersection of the curves

$$y = \cot \lambda L, \qquad y = -\frac{h_1}{\lambda l_1}.$$

A similar situation arises when $h_2 l_2 \neq 0$ and $l_1 = 0$.

Theorem 5.2 states that when a function $f(x)$ is piecewise smooth on the interval $0 \leq x \leq L$, we may write for $0 < x < L$

$$\frac{f(x+) + f(x-)}{2} = \sum_{n=1}^{\infty} c_n X_n(x) \qquad \text{where} \qquad c_n = \int_0^L f(x) X_n(x)\, dx. \quad (5.19)$$

Example 5.5 Expand the function $f(x) = 2x-1$, $0 \leq x \leq 4$ in terms of orthonormal eigenfunctions of the Sturm-Liouville system

$$X'' + \lambda^2 X = 0, \quad 0 < x < 4,$$
$$X'(0) = 0 = X(4).$$

Solution When we set $L = 4$ in line 6 of Table 5.1, normalized eigenfunctions of the Sturm-Liouville system are

$$X_n(x) = \frac{1}{\sqrt{2}} \cos \frac{(2n-1)\pi x}{8}, \quad n = 1, 2, \ldots$$

For $0 < x < 4$, we may write that $\quad 2x - 1 = \sum_{n=1}^{\infty} c_n X_n(x), \quad$ where

$$\begin{aligned} c_n &= \int_0^4 (2x-1) X_n(x)\, dx \\ &= \frac{1}{\sqrt{2}} \left\{ \frac{8(2x-1)}{(2n-1)\pi} \sin \frac{(2n-1)\pi x}{8} + \frac{128}{(2n-1)^2\pi^2} \cos \frac{(2n-1)\pi x}{8} \right\}_0^4 \\ &= \frac{-8[16 + 7(-1)^n(2n-1)\pi]}{\sqrt{2}(2n-1)^2\pi^2}. \end{aligned}$$

Thus,

$$\begin{aligned} 2x - 1 &= \sum_{n=1}^{\infty} \frac{-8[16 + 7(-1)^n(2n-1)\pi]}{\sqrt{2}(2n-1)^2\pi^2} \frac{1}{\sqrt{2}} \cos \frac{(2n-1)\pi x}{8} \\ &= -\frac{4\sqrt{2}}{\pi^2} \sum_{n=1}^{\infty} \frac{16 + 7(-1)^n(2n-1)\pi}{(2n-1)^2} \frac{1}{\sqrt{2}} \cos \frac{(2n-1)\pi x}{8}, \quad 0 < x < 4. \end{aligned}$$

Figure 5.5 shows a few partial sums of the series to illustrate convergence of the series to $2x-1$. Slowness of convergence is the result of the term $7\pi(-1)^n/(2n-1)$.●

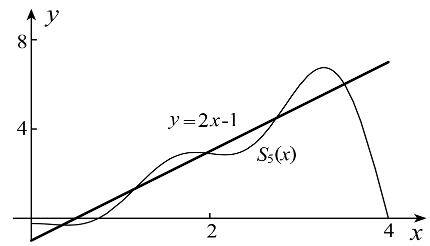

Figure 5.5a

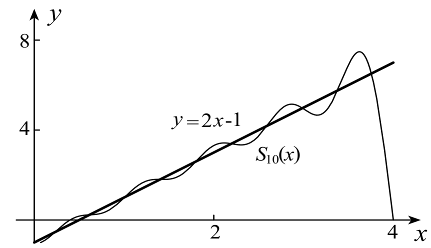

Figure 5.5b

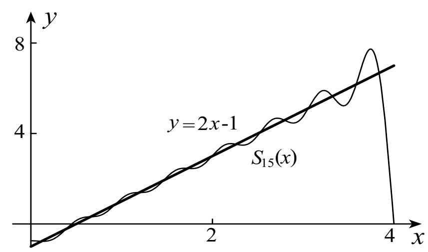

Figure 5.5c

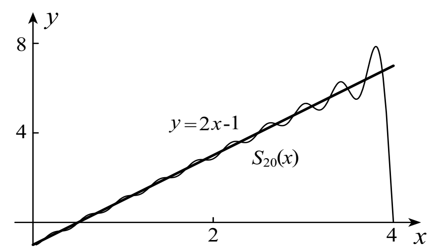

Figure 5.5d

Figure 5.5

Periodic Sturm-Liouville systems do not come under the purview of Theorem 5.2. In particular, this theorem does not guarantee expansions in terms of normalized eigenfunctions of periodic Sturm-Liouville systems. For instance, eigenvalues for the periodic Sturm-Liouville system of Example 5.2 are $\lambda_n = n^2\pi^2/L^2$ $(n = 0, 1, 2, \ldots)$, with corresponding eigenfunctions

$$\lambda_0 \leftrightarrow 1, \qquad \lambda_n \leftrightarrow \sin\frac{n\pi x}{L},\ \cos\frac{n\pi x}{L} \quad (n > 0).$$

Normalized eigenfunctions are

$$\lambda_0 \leftrightarrow \frac{1}{\sqrt{2L}}, \qquad \lambda_n \leftrightarrow \frac{1}{\sqrt{L}}\sin\frac{n\pi x}{L},\ \frac{1}{\sqrt{L}}\cos\frac{n\pi x}{L} \quad (n > 0).$$

Theorem 5.2 does not ensure the expansion of a function $f(x)$ in terms of these eigenfunctions, but our theory of ordinary Fourier series does. These are precisely the basis functions for ordinary Fourier series, except for normalizing factors, so we may write

$$\frac{f(x+) + f(x-)}{2} = \frac{a_0}{\sqrt{2L}} + \sum_{n=1}^{\infty}\left(a_n\frac{1}{\sqrt{L}}\cos\frac{n\pi x}{L} + b_n\frac{1}{\sqrt{L}}\sin\frac{n\pi x}{L}\right), \quad (5.20a)$$

where

$$a_0 = \int_{-L}^{L} f(x)\left(\frac{1}{\sqrt{2L}}\right)dx, \qquad a_n = \int_{-L}^{L} f(x)\left(\frac{1}{\sqrt{L}}\cos\frac{n\pi x}{L}\right)dx, \quad (5.20b)$$

$$b_n = \int_{-L}^{L} f(x)\left(\frac{1}{\sqrt{L}}\sin\frac{n\pi x}{L}\right)dx. \quad (5.20c)$$

As a final consideration in this section, we show that the Sturm-Liouville systems in Table 5.1 arise when separation of variables is applied to (initial) boundary value problems involving the second-order PDE

$$\nabla^2 V = p\frac{\partial^2 V}{\partial t^2} + q\frac{\partial V}{\partial t} + sV, \tag{5.21}$$

where p, q, and s are constants, t is time, and the Laplacian is expressed in Cartesian coordinates. We consider this PDE because it includes as special cases many of those in Chapter 2. In particular,

1. if $V = V(\mathbf{r}, t)$, $p = s = 0$, and $q = k^{-1}$, then 5.21 is the one-, two-, or three-dimensional heat conduction equation;
2. if $V = V(\mathbf{r}, t)$, $p = \rho/\tau$ (or ρ/E), then 5.21 is the one-, two-, or three-dimensional wave equation;
3. if $V = V(\mathbf{r})$, $p = q = s = 0$, then 5.21 is the one-, two-, or three-dimensional Laplace equation.

Thus, the results obtained here are valid for heat conduction, vibration, and potential problems, problems that we discuss in detail in Chapter 6.

When PDE 5.21 is to be solved in some finite region, boundary conditions and possibly initial conditions are associated with the PDE. If this region is a rectangular parallelopiped (box) in space, Cartesian coordinates can be chosen to specify the region in the form $0 \le x \le L$, $0 \le y \le L'$, $0 \le z \le L''$ (Figure 5.6). Boundary conditions must then be specified on the six faces. Suppose, for example, that the following homogeneous Dirichlet, Neumann, and Robin conditions accompany equation 5.21:

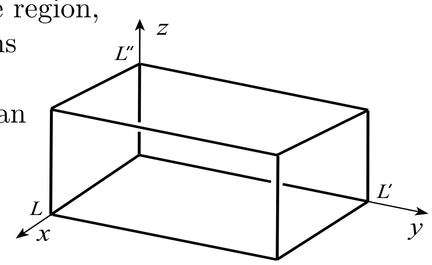

Figure 5.6

$$\nabla^2 V = p\frac{\partial^2 V}{\partial t^2} + q\frac{\partial V}{\partial t} + sV, \quad 0 < x < L, \quad 0 < y < L', \quad 0 < z < L'', \quad t > 0, \tag{5.22a}$$

$$V = 0, \quad x = 0, \quad 0 < y < L', \quad 0 < z < L'', \quad t > 0, \tag{5.22b}$$

$$\frac{\partial V}{\partial x} = 0, \quad x = L, \quad 0 < y < L', \quad 0 < z < L'', \quad t > 0, \tag{5.22c}$$

$$-l_3\frac{\partial V}{\partial y} + h_3 V = 0, \quad y = 0, \quad 0 < x < L, \quad 0 < z < L'', \quad t > 0, \tag{5.22d}$$

$$V = 0, \quad y = L', \quad 0 < x < L, \quad 0 < z < L'', \quad t > 0, \tag{5.22e}$$

$$\frac{\partial V}{\partial z} = 0, \quad z = 0, \quad 0 < x < L, \quad 0 < y < L', \quad t > 0, \tag{5.22f}$$

$$l_6\frac{\partial V}{\partial z} + h_6 V = 0, \quad z = L'', \quad 0 < x < L, \quad 0 < y < L', \quad t > 0, \tag{5.22g}$$

Initial conditions, if applicable. (5.22h)

If we assume that a function $V(x, y, z, t) = X(x)Y(y)Z(z)T(t)$ with variables separated satisfies PDE 5.22a,

$$X''YZT + XY''ZT + XYZ''T = pXYZT'' + qXYZT' + sXYZT.$$

Division by $XYZT$ gives

$$\frac{X''}{X} + \frac{Y''}{Y} + \frac{Z''}{Z} = \frac{pT'' + qT' + sT}{T},$$

or,

$$-\frac{X''}{X} = \frac{Y''}{Y} + \frac{Z''}{Z} - \frac{pT'' + qT' + sT}{T}.$$

The separation principle (see Section 4.1) implies that each side of this equation must be equal to a constant, say α:

$$-\frac{X''}{X} = \alpha = \frac{Y''}{Y} + \frac{Z''}{Z} - \frac{pT'' + qT' + sT}{T}. \tag{5.23}$$

Thus, $X(x)$ must satisfy the ODE $X'' + \alpha X = 0$, $0 < x < L$. When the separated function is substituted into boundary conditions 5.22b,c,

$$X(0)Y(y)Z(z)T(t) = 0, \qquad X'(L)Y(y)Z(z)T(t) = 0.$$

From these, $X(0) = 0 = X'(L)$, and hence $X(x)$ must satisfy

$$X'' + \alpha X = 0, \quad 0 < x < L, \tag{5.24a}$$

$$X(0) = 0 = X'(L). \tag{5.24b}$$

This is proper Sturm-Liouville system 5.14 with $l_1 = h_2 = 0$ and $h_1 = l_2 = 1$. When we set $\alpha = \lambda^2$, eigenvalues λ_n^2 and orthonormal eigenfunctions $X_n(x)$ are then given in line 8 of Table 5.1

$$\lambda_n^2 = \frac{(2n-1)^2\pi^2}{4L^2}, \qquad X_n(x) = \sqrt{\frac{2}{L}} \sin\frac{(2n-1)\pi x}{2L}.$$

Further separation of equation 5.23 gives

$$-\frac{Y''}{Y} = \beta = \frac{Z''}{Z} - \frac{pT'' + qT' + sT}{T} - \lambda_n^2, \tag{5.25}$$

where β is a constant. Boundary conditions 5.22d,e imply that $Y(y)$ must satisfy

$$Y'' + \beta Y = 0, \quad 0 < y < L', \tag{5.26a}$$

$$-l_3 Y'(0) + h_3 Y(0) = 0, \tag{5.26b}$$

$$Y(L') = 0. \tag{5.26c}$$

This is Sturm-Liouville system 5.14 with y's replacing x's, with h_3, l_3, and L' replacing h_1, l_1, and L, and with $l_2 = 0$ and $h_2 = 1$. When we set $\beta = \mu^2$, the eigenvalue equation and orthonormal eigenfunctions are found in line 3 of Table 5.1,

$$\cot\mu L' = -\frac{h_3}{\mu l_3}, \quad NY_m(y) = \frac{1}{\sin\mu_m L'}\sin\mu_m(L'-y), \quad 2N^2 = L'\left[1 + \left(\frac{h_3}{\mu_m l_3}\right)^2\right] + \frac{h_3/l_3}{\mu_m^2}.$$

Continued separation of equation 5.25 yields

$$-\frac{Z''}{Z} = \gamma = -\frac{pT'' + qT' + sT}{T} - \lambda_n^2 - \mu_m^2, \tag{5.27}$$

where γ is a constant. When this is combined with boundary conditions 5.22f,g, $Z(z)$ must satisfy the Sturm-Liouville system

$$Z'' + \gamma Z = 0, \quad 0 < z < L'', \tag{5.28a}$$
$$Z'(0) = 0, \tag{5.28b}$$
$$l_6 Z'(L'') + h_6 Z(L'') = 0. \tag{5.28c}$$

With changes in notation, this is the Sturm-Liouville system in line 4 of Table 5.1. Eigenvalues $\gamma = \nu^2$ are defined by

$$\tan \nu L'' = \frac{h_6}{\nu l_6},$$

with orthonormal eigenfunctions

$$\frac{1}{N} \cos \nu_j z \qquad \text{where} \qquad 2N^2 = L'' + \frac{h_6/l_6}{\nu_j^2 + (h_6/l_6)^2}.$$

The time-dependent part $T(t)$ of $V(x,y,z,t)$ is obtained from the ODE

$$pT'' + qT' + sT = -(\lambda_n^2 + \mu_m^2 + \nu_j^2)T. \tag{5.29}$$

In summary, separation of variables on (initial) boundary value problem 5.22 has led to the Sturm-Liouville systems in lines 3, 4, and 8 of Table 5.1. Other choices for boundary conditions lead to the remaining five Sturm-Liouville systems in Table 5.1 (see Exercises 30–32).

EXERCISES 5.2

1. Obtain expression 5.18c for $2N^2$ by direct integration of 5.18b. Hint: Show that

$$\sin \lambda_n L = \frac{(-1)^{n+1}\lambda_n \left(\dfrac{h_1}{l_1} + \dfrac{h_2}{l_2}\right)}{\left[\left(\lambda_n^2 + \dfrac{{h_1}^2}{{l_1}^2}\right)\left(\lambda_n^2 + \dfrac{{h_2}^2}{{l_2}^2}\right)\right]^{1/2}}, \qquad \cos \lambda_n L = \frac{(-1)^{n+1}\left(\lambda_n^2 - \dfrac{h_1 h_2}{l_1 l_2}\right)}{\left[\left(\lambda_n^2 + \dfrac{{h_1}^2}{{l_1}^2}\right)\left(\lambda_n^2 + \dfrac{{h_2}^2}{{l_2}^2}\right)\right]^{1/2}}.$$

2. For each Sturm-Liouville system in Table 5.1, find expressions for $\sin \lambda_n L$ and $\cos \lambda_n L$ that involve only h_1, h_2, l_1, l_2, and/or λ_n. These should be tabulated and attached to Table 5.1 for future reference.

3. Use equations 5.16 to verify that $\lambda = 0$ is an eigenvalue of Sturm-Liouville system 5.14 only when $h_1 = h_2 = 0$.

In Exercises 4–9 express the function $f(x) = x$, $0 \le x \le L$, in terms of orthonormal eigenfunctions of the Sturm-Liouville system. In the first four exercises, discuss convergence of the expansion at $x = 0$ and $x = L$.

4. $X'' + \lambda^2 X = 0, \quad X(0) = X(L) = 0$

5. $X'' + \lambda^2 X = 0, \quad X'(0) = X'(L) = 0$

6. $X'' + \lambda^2 X = 0, \quad X(0) = X'(L) = 0$

7. $X'' + \lambda^2 X = 0, \quad X'(0) = X(L) = 0$

8. $X'' + \lambda^2 X = 0, \quad X'(0) = 0, \quad l_2 X'(L) + h_2 X(L) = 0$

9. $X'' + \lambda^2 X = 0, \quad X(0) = 0, \quad l_2 X'(L) + h_2 X(L) = 0$

10. Express the function $f(x) = x^2$, $0 \le x \le L$, in terms of orthonormal eigenfunctions of the Sturm-Liouville system

$$X'' + \lambda^2 X = 0, \quad 0 < x < L,$$
$$X(0) = 0 = X'(L).$$

Does the expansion converge to $f(x)$ at $x = 0$ and $x = L$?

In Exercises 11–13 find eigenvalues and orthonormal eigenfunctions of the proper Sturm-Liouville sytem.

11. $\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} + \lambda^2 y = 0, \quad 0 < x < L, \quad y'(0) = 0 = y'(L)$

12. $\dfrac{d^2y}{dx^2} + \beta\dfrac{dy}{dx} + \lambda^2 y = 0, \quad 0 < x < L, \quad y(0) = 0 = y(L) \quad (\beta \neq 0$ a given constant)

13. $\dfrac{d^2y}{dx^2} + \beta\dfrac{dy}{dx} + \lambda^2 y = 0, \quad 0 < x < L, \quad y'(0) = 0 = y'(L) \quad (\beta \neq 0$ a given constant)

14. (a) Find eigenvalues and (nonnormalized) eigenfunctions for the proper Sturm-Liouville system

$$y'' + \lambda^2 y = 0, \quad -L < x < L,$$
$$y'(-L) = 0 = y'(L).$$

(b) Show that eigenfunctions in part (a) can be expressed in the compact form $\cos\dfrac{n\pi(x+L)}{2L}$, $n = 0, 1, 2, \ldots$.

(c) Normalize the eigenfunctions.

15. Find normalized eigenfunctions for the Sturm-Liouville system

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + \lambda^2 y = 0, \quad 1 < x < L,$$
$$y(1) = 0 = y(L).$$

Hint: Since the differential equation is of Cauchy-Euler type, set $y = x^m$.

16. Find normalized eigenfunctions of the Sturm-Liouville system in Exercise 15 if the boundary conditions are (a) $y'(1) = 0 = y(L)$ and (b) $y'(1) = 0 = y'(L)$.

17. On the basis of Exercises 15 and 16, we might be led to believe that eigenvalues and eigenfunctions of Sturm-Liouville systems associated with the differential equation in Exercise 15 on the interval $1 < x < L$, could be obtained by replacing x and L with $\ln x$ and $\ln L$ in Table 5.1. Show that this is not always the case by finding normalized eigenfunctions of the Sturm-Liouville system in Exercise 15 when boundary conditions are $y(1) = 0 = ly'(L) + hy(L)$.

18. Find nonnormalized eigenfunctions of the Sturm-Liouville system in Exercise 15 if the boundary conditions are $-l_1y'(1) + h_1y(1) = 0$ and $l_2y'(L) + h_2y(L) = 0$ with $h_1h_2l_1l_2 \neq 0$.

19. Find normalized eigenfunctions of the Sturm-Liouville system of Exercise 15 when the interval is $a \leq x \leq b$ and boundary conditions are (a) $y(a) = 0 = y(b)$, (b) $y'(a) = 0 = y(b)$, (c) $y'(a) = 0 = y'(b)$, and (d) $y(a) = 0 = ly'(b) + hy(b)$.

In Exercises 20–22 find six-figure approximations for the four smallest eigenvalues of the Sturm-Liouville system.

20. $X'' + \lambda^2 X = 0, \quad 0 < x < 1, \quad -X'(0) + 2000X(0) = 0, \quad X'(1) = 0$

21. $X'' + \lambda^2 X = 0, \quad 0 < x < 1, \quad X(0) = 0, \quad 3X'(1) + 2000X(1) = 0$

22. $X'' + \lambda^2 X = 0, \quad 0 < x < 1, \quad -X'(0) + 2X(0) = 0, \quad 2X'(1) + X(1) = 0$

23. (a) Expand the function

$$f(x) = \begin{cases} 1, & 0 < x < L/2 \\ -1, & L/2 < x < L \end{cases}$$

in terms of the normalized eigenfunctions of Sturm-Liouville system 5.2.
(b) What does the series converge to at $x = L/2$? Is this to be expected?
(c) What does the series converge to at $x = 0$ and $x = L$? Are these to be expected?

24. Repeat Exercise 23 with the eigenfunctions of Sturm-Liouville system 5.1.

25. In Exercise 11 of Section 5.1, we suggested two ways of interpreting the 4 in the differential equation. Does it make a difference as far as generalized Fourier series are concerned?

26. The initial boundary value problem for transverse vibrations $y(x,t)$ of a beam simply supported at one end ($x = L$) and horizontally built in at the other end ($x = 0$) when gravity is negligible compared with internal forces is

$$\begin{aligned} \frac{\partial^2 y}{\partial t^2} + c^2 \frac{\partial^4 y}{\partial x^4} &= 0, \quad 0 < x < L, \quad t > 0, \\ y(0,t) = y_x(0,t) &= 0, \quad t > 0, \\ y(L,t) = y_{xx}(L,t) &= 0, \quad t > 0, \\ y(x,0) &= f(x), \quad 0 < x < L, \\ y_t(x,0) &= g(x), \quad 0 < x < L. \end{aligned}$$

(a) Show that when $y(x,t)$ is set equal to $X(x)T(t)$, eigenfunctions obtained are

$$X_n(x) = \frac{1}{\cos \lambda_n L} \sin \lambda_n (L - x) - \frac{1}{\cosh \lambda_n L} \sinh \lambda_n (L - x),$$

where eigenvalues λ_n must satisfy

$$\tan \lambda L = \tanh \lambda L.$$

(b) Prove that these eigenfunctions are orthogonal on the interval $0 \le x \le L$ with respect to the weight function $p(x) = 1$. (Hint: Use the differential equation defining $X_n(x)$ and a construction like that in Theorem 5.1.)

27. Does the Sturm-Liouville system in line 6 of Table 5.1 give rise to the expansion in Exercise 21 of Section 3.2 for even and odd-harmonic functions?

28. Does the Sturm-Liouville system in line 8 of Table 5.1 give rise to the expansion in Exercise 20 of Section 3.2 for odd and odd-harmonic functions?

29. Show that the Sturm-Liouville system

$$\begin{aligned} \frac{d^2 X}{dx^2} + \lambda X &= 0, \quad 0 < x < L, \\ X'(0) &= 0, \\ l_2 X'(L) - h_2 X(L) &= 0, \quad (l_2 > 0, \ h_2 > 0) \end{aligned}$$

has exactly one negative eigenvalue. What is the corresponding eigenfunction?

In Exercises 30–32 determine all Sturm-Liouville systems that result when separation of variables is used to solve the problem. Do not solve the problem; simply find the Sturm-Liouville systems. Find eigenvalues (or eigenvalue equations) and orthonormal eigenfunctions for each Sturm-Liouville system. Give a physical interpretation of each problem.

30.

$$\begin{aligned}
\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} &= \frac{1}{k}\frac{\partial U}{\partial t}, \quad 0 < x < L, \quad 0 < y < L', \quad t > 0,\\
U(0, y, t) &= 0, \quad 0 < y < L', \quad t > 0,\\
\frac{\partial U(L, y, t)}{\partial x} + 200U(L, y, t) &= 0, \quad 0 < y < L', \quad t > 0,\\
\frac{\partial U(x, 0, t)}{\partial y} &= 0, \quad 0 < x < L, \quad t > 0,\\
\frac{\partial U(x, L', t)}{\partial y} &= 0, \quad 0 < x < L, \quad t > 0,\\
U(x, y, 0) &= f(x, y), \quad 0 < x < L, \quad 0 < y < L'.
\end{aligned}$$

31.

$$\begin{aligned}
\frac{\partial^2 y}{\partial t^2} &= c^2\frac{\partial^2 y}{\partial x^2} - \beta\frac{\partial y}{\partial t}, \quad 0 < x < L, \quad t > 0,\\
-\tau\frac{\partial y(0, t)}{\partial x} + ky(0, t) &= 0, \quad t > 0,\\
y(L, t) &= 0, \quad t > 0,\\
y(x, 0) &= f(x), \quad 0 < x < L,\\
y_t(x, 0) &= 0, \quad 0 < x < L.
\end{aligned}$$

32.

$$\begin{aligned}
\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} &= 0, \quad 0 < x < L, \quad 0 < y < L', \quad 0 < z < L'',\\
V(0, y, z) &= 0, \quad 0 < y < L', \quad 0 < z < L'',\\
\frac{\partial V(L, y, z)}{\partial x} &= 0, \quad 0 < y < L', \quad 0 < z < L'',\\
V(x, 0, z) &= 0, \quad 0 < x < L, \quad 0 < z < L'',\\
V(x, L', z) &= 0, \quad 0 < x < L, \quad 0 < z < L'',\\
V(x, y, 0) &= f(x, y), \quad 0 < x < L, \quad 0 < y < L',\\
V(x, y, L'') &= 0, \quad 0 < x < L, \quad 0 < y < L'.
\end{aligned}$$

33. A fourth-order Sturm-Liouville system consists of a fourth-order, homogeneous differential equation of the following form, together with four linear, homogeneous boundary conditions for a function $y(\lambda, x)$:

$$\begin{aligned}
\frac{d^2}{dx^2}\left[r(x)\frac{d^2 y}{dx^2}\right] + [\lambda p(x) - q(x)]y &= 0, \quad a < x < b,\\
l_1(ry'')' + h_1 y &= 0, \quad x = a,\\
l_2(ry'') + h_2 y' &= 0, \quad x = a,\\
l_3(ry'')' + h_3 y &= 0, \quad x = b,\\
l_4(ry'') + h_4 y' &= 0, \quad x = b,
\end{aligned}$$

where $p(x)$, $q(x)$, and $r''(x)$ are continuous on $a \le x \le b$, and $p > 0$ and $r > 0$ for $a \le x \le b$. Assuming that the system has eigenfunctions, show that eigenfunctions corresponding to distinct eigenvalues are orthogonal on $a \le x \le b$ with respect to the weight function $p(x)$.

34. Show that when separation of variables is applied to the homogeneous beam equation 2.95 and boundary conditions corresponding to simple supports, ends built-in horizontally, and/or cantilevered ends, the Sturm-Liouville system of Exercise 33 results.