§13.2 Green's Functions for Dirichlet Boundary Value Problems

Dirichlet problems for the two-dimensional Helmholtz equation take the form

$$Lu = \nabla^2 u + k^2 u = F(x,y), \quad (x,y) \text{ in } A, \tag{13.15a}$$

$$u(x,y) = K(x,y), \quad (x,y) \text{ on } \beta(A). \tag{13.15b}$$

For $k = 0$, we have the special case of Poisson's equation. When $F(x,y)$ has continuous first derivatives and piecewise continuous second derivatives in A, as does $K(x,y)$ on $\beta(A)$, this problem has a unique solution. The special case in which A is a rectangle was discussed in Section 6.7 (see problem 6.70). In practical situations when $F(x,y)$ and $K(x,y)$ may not satisfy these conditions, verification of uniqueness is much more difficult, as is finding the solution by previous methods. Green's functions provide an excellent alternative.

We define the Green's function $G(x,y;X,Y)$ for problem 13.15 as the solution of

$$LG = \nabla^2 G + k^2 G = \delta(x-X, y-Y), \quad (x,y) \text{ in } A, \tag{13.16a}$$

$$G(x,y;X,Y) = 0, \quad (x,y) \text{ on } \beta(A). \tag{13.16b}$$

It is the solution of problem 13.15 due to a unit source at the point (X,Y) when boundary conditions are homogeneous. In Section 13.3, we shall prove that the solution of boundary value problem 13.15 can be expressed in the form

$$u(x,y) = \iint_A G(x,y;X,Y)F(X,Y)\,dA + \oint_{\beta(A)} K(X,Y)\frac{\partial G(x,y;X,Y)}{\partial N}ds, \tag{13.17}$$

where $\partial G/\partial N$ is the outward normal derivative of G with respect to the (X,Y) variables along $\beta(A)$. The solution is expressed in terms of integrals of the associated Green's function and source and boundary terms $F(x,y)$ and $K(x,y)$. We shall also interpret these integrals physically. In this section, we concentrate on methods for finding Green's functions.

If we substitute $u = G(x,y;X,Y)$ and $v = G(x,y;R,S)$ into Green's identity 13.14a,

$$\iint_A [G(x,y;R,S)\nabla^2 G(x,y;X,Y) - G(x,y;X,Y)\nabla^2 G(x,y;R,S)]\,dA = 0$$

(since $G(x,y;R,S)$ and $G(x,y;X,Y)$ satisfy boundary condition13.16b). But because G is a solution of PDE 13.16a, we may write

$$\begin{aligned} 0 &= \iint_A \{G(x,y;R,S)[\delta(x-X,y-Y) - k^2 G(x,y;X,Y)] \\ &\qquad - G(x,y;X,Y)[\delta(x-R,y-S) - k^2 G(x,y;R,S)]\}\,dA \\ &= G(X,Y;R,S) - G(R,S;X,Y). \end{aligned}$$

In other words, the Green's function is symmetric under the interchange of first and second variables with third and fourth,

$$G(x,y;X,Y) = G(X,Y;x,y). \tag{13.18}$$

This result is also valid when boundary condition 13.15b is replaced by either a Neumann or a Robin condition.

For boundary value problems associated with ODEs, we derived general formulas (equations 12.33 and 12.34 in Section 12.3) for Green's functions. This was possible because boundaries for ODEs consist of two points. For PDEs, boundaries consist of curves for two-dimensional problems and surfaces for three-dimensional problems. As a result, it is impossible to find formulas for Green's functions associated with multivariable boundary value problems. What we can do is develop general techniques useful in large classes of problems. In this section, we illustrate four of these techniques for finding the Green's function for Dirichlet problem 13.15 in the case of Poisson's equation,

$$Lu = \nabla^2 u = F(x,y), \quad (x,y) \text{ in } A, \tag{13.19a}$$

$$u(x,y) = K(x,y), \quad (x,y) \text{ on } \beta(A). \tag{13.19b}$$

The Green's function for this problem satisfies

$$LG = \nabla^2 G = \delta(x - X, y - Y), \quad (x,y) \text{ in } A, \tag{13.20a}$$

$$G(x,y;X,Y) = 0, \quad (x,y) \text{ on } \beta(A). \tag{13.20b}$$

These techniques may also be appropriate for boundary value problems with Neumann or Robin conditions or mixed problems (problems with different types of boundary conditions on different parts of the boundary), and also for the Helmholtz equation.

Full Eigenfunction Expansion

In this method, the Green's function is expanded in terms of orthonormal eigenfunctions of the associated eigenvalue problem

$$Lu + \lambda^2 u = 0, \quad (x,y) \text{ in } A, \tag{13.21a}$$

$$u(x,y) = 0, \quad (x,y) \text{ on } \beta(A). \tag{13.21b}$$

We illustrate with the following example.

Example 13.1 Find the Green's function associated with the Dirichlet problem for the two-dimensional Laplacian on a rectangle A: $0 \le x \le L$, $0 \le y \le L'$.

Solution Separation of variables on

$$\nabla^2 u + \lambda^2 u = 0, \quad (x,y) \text{ in } A, \tag{13.22a}$$

$$u(x,y) = 0, \quad (x,y) \text{ on } \beta(A), \tag{13.22b}$$

leads to normalized eigenfunctions

$$u_{mn}(x,y) = \frac{2}{\sqrt{LL'}} \sin\frac{n\pi x}{L} \sin\frac{m\pi y}{L'},$$

corresponding to eigenvalues $\lambda_{mn}^2 = (n\pi/L)^2 + (m\pi/L')^2$ (see Section 6.5). The eigenfunction expansion of $G(x,y;X,Y)$ in terms of these eigenfunctions is

$$G(x,y;X,Y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} c_{mn} u_{mn}(x,y), \tag{13.23}$$

and this representation satisfies the boundary condition that G vanish on the edges of the rectangle. To calculate the coefficients c_{mn}, we substitute this representation into the PDE $\nabla^2 G = \delta(x-X, y-Y)$ for G and expand the delta function in terms of the $u_{mn}(x,y)$,

$$\begin{aligned}
\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} & c_{mn}\left(-\frac{n^2\pi^2}{L^2} - \frac{m^2\pi^2}{L'^2}\right) u_{mn}(x,y) \\
&= \delta(x-X, y-Y) \\
&= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left[\int_0^L\int_0^{L'} \delta(x-X, y-Y) u_{mn}(x,y)\, dy\, dx\right] u_{mn}(x,y) \\
&= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} u_{mn}(X,Y) u_{mn}(x,y).
\end{aligned}$$

Consequently, $c_{mn} = u_{mn}(X,Y)/(-\lambda_{mn}^2)$, and

$$\begin{aligned}
G(x,y;X,Y) &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{u_{mn}(X,Y)}{-\lambda_{mn}^2} u_{mn}(x,y) \\
&= \frac{-4}{LL'}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{1}{\left(\frac{n\pi}{L}\right)^2 + \left(\frac{m\pi}{L'}\right)^2} \sin\frac{n\pi X}{L}\sin\frac{m\pi Y}{L'}\sin\frac{n\pi x}{L}\sin\frac{m\pi y}{L'}. \quad (13.24)
\end{aligned}$$

In Exercise 1 it is shown that this representation can also be obtained using Green's identity 13.14a. This avoids the interchange of the Laplacian and summation operations and the eigenfunction expansion of $\delta(x-X, y-Y)$.•

A general formula for full eigenfunction expansions can be found in Exercise 2, but such representations are of limited calculational utility. First, they are possible only when the eigenvalue problem can be separated, and this requires that the boundary of A consist of coordinate curves (or coordinate surfaces, in three-dimensional problems), Second, in the case in which the full eigenfunction expansion is available, a partial eigenfunction expansion that converges more rapidly is also available.

Partial Eigenfunction Expansion

Like the full eigenfunction expanison, this method requires that region A be bounded by coordinate curves (or coordinate surfaces, in three-dimensional problems). It differs in that separation is considered on the homogeneous problem

$$Lu = 0, \quad (x,y) \text{ in } A, \qquad (13.25a)$$

$$u(x,y) = 0, \quad (x,y) \text{ on } \beta(A), \qquad (13.25b)$$

and is carried out until one variable remains. An eigenfunction expansion for the Green's function is then found in terms of normalized eigenfunctions already determined, with coefficients that are functions of the remaining variable. We illustrate once again with the problem in Example 13.1.

Example 13.2 Find a partial eigenfunction representation for the Green's function in Example 13.1.

Solution Separation of variables on

$$\nabla^2 u = 0, \quad (x, y) \text{ in } A, \tag{13.26a}$$
$$u(x, y) = 0, \quad (x, y) \text{ on } \beta(A), \tag{13.26b}$$

leads to normalized eigenfunctions $f_n(x) = \sqrt{\dfrac{2}{L}} \sin \dfrac{n\pi x}{L}$. We expand $G(x, y; X, Y)$ in terms of these,

$$G(x, y; X, Y) = \sum_{n=1}^{\infty} a_n(y) f_n(x). \tag{13.27}$$

In actual fact, coefficients $a_n(y)$ must also be functions of X and Y, but we shall understand this dependence implicitly rather than express it explicitly. To determine the $a_n(y)$, we substitute this expression into the PDE $\nabla^2 G = \delta(x - X, y - Y)$ for G and expand the delta function in terms of the $f_n(x)$,

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{-n^2\pi^2}{L^2} a_n f_n(x) + \sum_{n=1}^{\infty} \frac{d^2 a_n}{dy^2} f_n(x) &= \delta(x - X, y - Y) \\
&= \sum_{n=1}^{\infty} \left[\int_0^L \delta(x - X, y - Y) f_n(x)\, dx \right] f_n(x) \\
&= \sum_{n=1}^{\infty} f_n(X) \delta(y - Y) f_n(x).
\end{aligned}$$

This equation and the boundary conditions $G(x, 0; X, Y) = 0 = G(x, L'; X, Y)$ require the $a_n(y)$ to satisfy

$$\frac{d^2 a_n}{dy^2} - \frac{n^2\pi^2}{L^2} a_n = \delta(y - Y) f_n(X), \quad 0 < y < L',$$
$$a_n(0) = 0, \quad a_n(L') = 0.$$

We can solve this boundary value problem most easily by using our theory of Green's functions for ODEs. Since a solution of the homogeneous equation that satisfies the first boundary condition is $\sinh(n\pi y/L)$, and one that satisfies the second is $\sinh[n\pi(L' - y)/L]$, equation 12.34 in Section 12.3 gives

$$a_n(y) = \frac{1}{J} \left[\sinh \frac{n\pi y}{L} \sinh \frac{n\pi(L' - Y)}{L} h(Y - y) + \sinh \frac{n\pi Y}{L} \sinh \frac{n\pi(L' - y)}{L} h(y - Y) \right],$$

where J is the conjunct of $\sinh(n\pi y/L)$ and $\sinh[n\pi(L' - y)/L]$,

$$\begin{aligned}
J &= \frac{1}{f_n(X)} \left[\sinh \frac{n\pi y}{L} \left(\frac{-n\pi}{L} \right) \cosh \frac{n\pi(L' - y)}{L} - \left(\frac{n\pi}{L} \right) \cosh \frac{n\pi y}{L} \sinh \frac{n\pi(L' - y)}{L} \right] \\
&= -\frac{n\pi \sinh(n\pi L'/L)}{\sqrt{2L} \sin(n\pi X/L)}.
\end{aligned}$$

Thus, an alternative to the double-series, full eigenfunction expansion is the single-series, partial eigenfunction expansion

$$G(x,y;X,Y) = \sum_{n=1}^{\infty} \frac{-\sqrt{2}L \sin \frac{n\pi X}{L}}{n\pi \sinh \frac{n\pi L'}{L}} \left[\sinh \frac{n\pi y}{L} \sinh \frac{n\pi(L'-Y)}{L} h(Y-y) \right.$$

$$\left. + \sinh \frac{n\pi Y}{L} \sinh \frac{n\pi(L'-y)}{L} h(y-Y) \right] \sqrt{\frac{2}{L}} \sin \frac{n\pi x}{L}$$

$$= \begin{cases} \displaystyle\sum_{n=1}^{\infty} \frac{-2 \sin \frac{n\pi X}{L} \sin \frac{n\pi x}{L} \sinh \frac{n\pi y}{L} \sinh \frac{n\pi(L'-Y)}{L}}{n\pi \sinh \frac{n\pi L'}{L}}, & 0 \le y \le Y \\ \displaystyle\sum_{n=1}^{\infty} \frac{-2 \sin \frac{n\pi X}{L} \sin \frac{n\pi x}{L} \sinh \frac{n\pi Y}{L} \sinh \frac{n\pi(L'-y)}{L}}{n\pi \sinh \frac{n\pi L'}{L}}, & Y \le y \le L'. \end{cases} \tag{13.28}$$

It is clear that we could find an equivalent solution by expanding G in a Fourier sine series in y. The result would be

$$G(x,y;X,Y) = \begin{cases} \displaystyle\sum_{n=1}^{\infty} \frac{-2 \sin \frac{n\pi Y}{L'} \sin \frac{n\pi y}{L'} \sinh \frac{n\pi x}{L'} \sinh \frac{n\pi(L-X)}{L'}}{n\pi \sinh \frac{n\pi L}{L'}}, & 0 \le x \le X \\ \displaystyle\sum_{n=1}^{\infty} \frac{-2 \sin \frac{n\pi Y}{L'} \sin \frac{n\pi y}{L'} \sinh \frac{n\pi X}{L'} \sinh \frac{n\pi(L-x)}{L'}}{n\pi \sinh \frac{n\pi L}{L'}}, & X \le x \le L. \end{cases} \tag{13.29}$$

A natural question to ask is: In which problems, should each of these expressions for $G(x,y;X,Y)$ be used? Since each is a Fourier series, rates of convergence of the series will depend on the relative magnitudes of coefficients. The coefficient of $\sin(n\pi x/L)$ in representation 13.28 for $y > Y$ is

$$\frac{-2 \sin \frac{n\pi X}{L} \sinh \frac{n\pi Y}{L} \sinh \frac{n\pi(L'-y)}{L}}{n\pi \sinh \frac{n\pi L'}{L}},$$

and for large n we may drop the negative exponentials in the hyperbolic functions and approximate this quantity with

$$-\frac{e^{n\pi Y/L} e^{n\pi(L'-y)/L}}{n\pi e^{n\pi L'/L}} \sin \frac{n\pi X}{L} = \frac{-1}{n\pi} e^{n\pi(Y-y)/L} \sin \frac{n\pi X}{L}.$$

Similarly, when $y < Y$, the coefficient can, for large n, be approximated by

$$\frac{-1}{n\pi} e^{n\pi(y-Y)/L} \sin \frac{n\pi X}{L}.$$

Corresponding coefficients in representation 13.29 are approximated for large n by

$$\frac{-1}{n\pi} e^{n\pi|X-x|/L'} \sin \frac{n\pi Y}{L'}.$$

It follows that to calculate $G(x,y;X,Y)$ at a value of x that is substantially different from X, it would be wise to use representation 13.29, and, conversely, when y is markedly different from Y, representation 13.28 would provide faster convergence.

In addition, when boundary integrals arise for the solution of Dirichlet problem 13.19 (and this occurs for nonhomogeneous boundary conditions 13.19b), it is advantageous to use representation 13.28 for integrations along $y = 0$ and $y = L'$, but use representation 13.29 along $x = 0$ and $x = L$.

Splitting Technique

Sometimes it is convenient to split G into two parts, $G = U+g$, where U contains the singular part of G due to the delta function in PDE 13.20a and g guarantees that G satisfies the boundary conditions associated with L. This splitting technique permits consideration of the singular nature of the Green's function without the annoyance of boundary conditions. (The technique could have been used for ODEs, but it was unnecessary because formulas 12.33 and 12.34 in Section 12.3 were presented for Green's functions.) To be more specific, for the Green's function satisfying problem 13.20, we set $G = U + g$, where $U(x, y; X, Y)$ satisfies the PDE

$$LU = \delta(x - X, y - Y) \tag{13.30}$$

and g satisfies the boundary value problem

$$Lg = 0, \quad (x, y) \text{ in } A, \tag{13.31a}$$

$$g = -U, \quad (x, y) \text{ on } \beta(A). \tag{13.31b}$$

Because $U(x, y; X, Y)$ is not required to satisfy boundary conditions, it is often called the **free-space Green's function for the operator** L. Free-space Green's functions for the Laplace and Helmholtz operators in two and three dimensions are listed in Table 13.1. Each is singular at the source point (X, Y).

	∇^2 Laplacian	$\nabla^2 + k^2$ Helmholtz
xy plane	$\dfrac{1}{2\pi} \ln \sqrt{(x-X)^2 + (y-Y)^2}$	$\dfrac{1}{4} Y_0[k\sqrt{(x-X)^2 + (y-Y)^2}]$
xyz space	$\dfrac{-1}{4\pi\sqrt{(x-X)^2 + (y-Y)^2 + (z-Z)^2}}$	$-\dfrac{e^{ik\sqrt{(x-X)^2+(y-Y)^2+(z-Z)^2}}}{4\pi\sqrt{(x-X)^2 + (y-Y)^2 + (z-Z)^2}}$, $-\dfrac{e^{-ik\sqrt{(x-X)^2+(y-Y)^2+(z-Z)^2}}}{4\pi\sqrt{(x-X)^2 + (y-Y)^2 + (z-Z)^2}}$

Table 13.1

We justify the first entry here; the other three are discussed in the exercises. The two-space Green's function $G(x, y; X, Y)$ for the Laplacian is the solution of

$$\nabla^2 G = \delta(x - X, y - Y).$$

It is the effect at point (x, y) due to a unit source at (X, Y). Because the function should be symmetric about the source point, we switch to polar coordinates centred at (X, Y), and search for a function $G(r; 0)$ satisfying

$$\frac{d^2G}{dr^2} + \frac{1}{r}\frac{dG}{dr} = \frac{\delta(r)}{2\pi r}, \tag{13.32}$$

where we have used equation 13.9 for the delta function at the origin. Multiplication by r leads to

$$\frac{d}{dr}\left(r\frac{dG}{dr}\right) = \frac{\delta(r)}{2\pi}.$$

Integration with respect to r from $r = 0$ to an arbitrary value of r gives

$$r\frac{dG}{dr} = \frac{1}{2\pi} \quad \Longrightarrow \quad \frac{dG}{dr} = \frac{1}{2\pi r} \quad \Longrightarrow \quad G(r;0) = \frac{1}{2\pi}\ln r + C.$$

We take $C = 0$. This shows that the effect at a point due to a unit source is $1/(2\pi)$ times the logarithm of the distance from point to source. It follows that the effect at point (x, y) due to a source at (X, Y) is

$$G(x, y; X, Y) = \frac{1}{2\pi}\ln\sqrt{(x-X)^2 + (y-Y)^2}.$$

A similar derivation gives the free-space Green's function for the three-dimensional Laplacian (Exercise 25). Unfortunately, the same technique does not work for the Helmholtz operator. In Exercise 26, we provide an alternative derivation for free-space Green's functions associated with the Laplacian and this technique does extend to Helmholtz operators (Exercises 28 and 29).

We now return to the splitting technique by illustrating it in the following example.

Example 13.3 Find the Green's function for the Dirichlet problem associated with Laplace's equation on a circle $0 \le r \le a$.

Solution The Green's function associated with the Dirichlet problem for the Laplacian on a circle centred at the origin with radius a satisfies

$$\nabla^2 G = \frac{\delta(r-R)\delta(\theta-\Theta)}{r}, \quad 0 < r < a, \quad -\pi < \theta \le \pi, \quad (13.33a)$$

$$G(a, \theta; R, \Theta) = 0, \quad -\pi < \theta \le \pi. \quad (13.33b)$$

The free-space Green's function for the two-dimensional Laplacian with singularity at (R, Θ) is

$$\begin{aligned} U(r, \theta; R, \Theta) &= \frac{1}{2\pi}\ln\sqrt{(r\cos\theta - R\cos\Theta)^2 + (r\sin\theta - R\sin\Theta)^2} \\ &= \frac{1}{4\pi}\ln\left[r^2 + R^2 - 2rR\cos(\theta-\Theta)\right] \end{aligned}$$

(see Table 13.1). When we split G into $G = U + g$, function g must satisfy

$$\nabla^2 g = 0, \quad 0 < r < a, \quad -\pi < \theta \le \pi, \quad (13.34a)$$

$$g(a, \theta; R, \Theta) = -\frac{1}{4\pi}\ln\left[a^2 + R^2 - 2aR\cos(\theta-\Theta)\right], \quad -\pi < \theta \le \pi. \quad (13.34b)$$

Separation of variables on the PDE, together with boundedness at $r = 0$, leads to a solution of the form

$$g(r, \theta; R, \Theta) = \frac{A_0}{\sqrt{2\pi}} + \sum_{n=1}^{\infty}\left(A_n r^n \frac{\cos n\theta}{\sqrt{\pi}} + B_n r^n \frac{\sin n\theta}{\sqrt{\pi}}\right)$$

(see equation 6.31a in Section 6.3). Boundary condition 13.34b requires

$$\frac{A_0}{\sqrt{2\pi}}+\sum_{n=1}^{\infty}\left(A_n a^n\frac{\cos n\theta}{\sqrt{\pi}}+B_n a^n\frac{\sin n\theta}{\sqrt{\pi}}\right)=\frac{-1}{4\pi}\ln\left[a^2+R^2-2aR\cos(\theta-\Theta)\right]$$
$$=\frac{-1}{4\pi}\ln a^2-\frac{1}{4\pi}\ln\left[1+\left(\frac{R}{a}\right)^2-2\left(\frac{R}{a}\right)\cos(\theta-\Theta)\right].$$

With the result

$$\sum_{n=1}^{\infty}\frac{\alpha^n\cos n\phi}{n}=-\frac{1}{2}\ln(1+\alpha^2-2\alpha\cos\phi),\quad (|\alpha|<1), \tag{13.35}$$

we may write

$$\frac{A_0}{\sqrt{2\pi}}+\sum_{n=1}^{\infty}\left(A_n a^n\frac{\cos n\theta}{\sqrt{\pi}}+B_n a^n\frac{\sin n\theta}{\sqrt{\pi}}\right)$$
$$=\frac{-1}{4\pi}\ln a^2+\frac{1}{2\pi}\sum_{n=1}^{\infty}\frac{(R/a)^n}{n}\cos n(\theta-\Theta)$$
$$=\frac{-1}{4\pi}\ln a^2+\frac{1}{2\pi}\sum_{n=1}^{\infty}\frac{(R/a)^n}{n}(\cos n\theta\cos n\Theta+\sin n\theta\sin n\Theta).$$

Comparison of coefficients requires

$$\frac{A_0}{\sqrt{2\pi}}=\frac{-1}{4\pi}\ln a^2,\qquad \frac{A_n a^n}{\sqrt{\pi}}=\frac{(R/a)^n}{2\pi n}\cos n\Theta,\qquad \frac{B_n a^n}{\sqrt{\pi}}=\frac{(R/a)^n}{2\pi n}\sin n\Theta,$$

and therefore

$$g(r,\theta;R,\Theta)=\frac{-1}{2\pi}\ln a+\sum_{n=1}^{\infty}r^n\left[\frac{(R/a)^n}{2\pi n a^n}\cos n\theta\cos n\Theta+\frac{(R/a)^n}{2\pi n a^n}\sin n\theta\sin n\Theta\right]$$
$$=\frac{-1}{2\pi}\ln a+\frac{1}{2\pi}\sum_{n=1}^{\infty}\frac{(rR/a^2)^n}{n}\cos n(\theta-\Theta).$$

But identity 13.35 permits evaluation of this series in closed form,

$$g(r,\theta;R,\Theta)=\frac{-1}{2\pi}\ln a-\frac{1}{4\pi}\ln\left[1+\left(\frac{rR}{a^2}\right)^2-2\left(\frac{rR}{a^2}\right)\cos(\theta-\Theta)\right]$$
$$=\frac{1}{2\pi}\ln a-\frac{1}{4\pi}\ln\left[a^4+R^2r^2-2a^2Rr\cos(\theta-\Theta)\right].$$

Finally,

$$G(r,\theta;R,\Theta)=U+g=\frac{1}{4\pi}\ln\left[r^2+R^2-2Rr\cos(\theta-\Theta)\right]+\frac{1}{2\pi}\ln a$$
$$-\frac{1}{4\pi}\ln\left[a^4+R^2r^2-2a^2Rr\cos(\theta-\Theta)\right]$$
$$=\frac{1}{4\pi}\ln\left[a^2\frac{r^2+R^2-2Rr\cos(\theta-\Theta)}{a^4+R^2r^2-2a^2Rr\cos(\theta-\Theta)}\right]. \tag{13.36}$$

This result is also obtained with a partial eigenfunction expansion in Exercise 13.•

The splitting technique points out a distinct difference between Green's functions for one-dimensional problems and those for multidimensional problems. The Green's function $g(x;X)$ for a one-dimensional boundary value problem (associated with a second-order ODE) is a continuous function of x (or can be made so) with a jump discontinuity in its first derivative. Green's functions for multidimensional boundary value problems can always be represented as the sum of a free-space Green's function U and a regular part g, and, according to Table 13.1, free-space Green's functions are always singular at the source point. Thus, multivariable Green's functions always have discontinuities at source points.

Method of Images

The method of images is simply physical reasoning and intelligent guesswork in arriving at the function g in the splitting technique, and as such it works only on Laplace's equation with very simple geometries. When the Green's function G for a domain A is split into $U+g$, the free-space Green's function U can be regarded as the potential due to a unit point source interior to A. This source, by itself, induces a nonzero potential on $\beta(A)$. What is needed is a source distribution exterior to A whose potential g will cancel the effect of U on $\beta(A)$. (The fact that this distribution is exterior to A guarantees that $G=U+g$ satisfies $\nabla^2 G=\delta$ interior to A.)

We illustrate with the following three-dimensional problem.

Example 13.4 Find the Green's function associated with the three-dimensional Dirichlet problem for Laplace's equation in a sphere of radius a.

Solution The Green's function satisfies

$$\nabla^2 G = \frac{\delta(r-R)\delta(\theta-\Theta)\delta(\phi-\Phi)}{r^2\sin\phi}, \quad 0<r<a, \quad 0<\phi<\pi, \quad -\pi<\theta\le\pi, \tag{13.37a}$$

$$G(a,\phi,\theta;R,\Phi,\Theta)=0, \quad 0<\phi<\pi, \quad -\pi<\theta\le\pi. \tag{13.37b}$$

According to Table 13.1, the free-space Green's function with source point (X,Y,Z) is $-1/[4\pi\sqrt{(x-X)^2+(y-Y)^2+(z-Z)^2}]$. When (R,Φ,Θ) are the spherical coordinates of (X,Y,Z), this function becomes

$$U(r,\phi,\theta;R,\Phi,\Theta)=\frac{-1}{4\pi\sqrt{r^2+R^2-2Rr[\cos\phi\cos\Phi+\sin\phi\sin\Phi\cos(\theta-\Theta)]}}.$$

What the method of images suggests is finding a source distribution exterior to the sphere, the potential g for which is such that $G=U+g$ vanishes on $r=a$. We might first consider whether a single source of magnitude q at a point $(R^*,\Phi^*,\Theta^*)(R^*>a)$ might suffice. Symmetry would suggest that such a source could eliminate U on $r=a$, which is symmetric around the line through the origin, and (R,Φ,Θ)

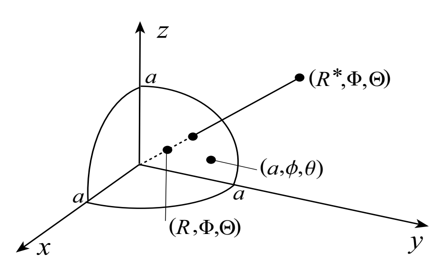

Figure 13.1

(Figure 13.1) only if (R^*, Φ^*, Θ^*) were to lie on the line also. We assume, therefore, that $\Theta^* = \Theta$ and $\Phi^* = \Phi$, in which case the condition that $G = U + g$ vanish on $r = a$ is

$$0 = \frac{-1}{4\pi\sqrt{a^2 + R^2 - 2aR[\cos\phi\cos\Phi + \sin\phi\sin\Phi\cos(\theta - \Theta)]}} + \frac{-q}{4\pi\sqrt{a^2 + {R^*}^2 - 2aR^*[\cos\phi\cos\Phi + \sin\phi\sin\Phi\cos(\theta - \Theta)]}}$$

or,

$$-q\sqrt{a^2 + R^2 - 2aR[\cos\phi\cos\Phi + \sin\phi\sin\Phi\cos(\theta - \Theta)]} = \sqrt{a^2 + {R^*}^2 - 2aR^*[\cos\phi\cos\Phi + \sin\phi\sin\Phi\cos(\theta - \Theta)]}.$$

Since this condition must be valid for all ϕ and θ, we set $\phi = 0$ and $\phi = \pi$,

$$-q\sqrt{a^2 + R^2 - 2aR\cos\Phi} = \sqrt{a^2 + {R^*}^2 - 2aR^*\cos\Phi},$$
$$-q\sqrt{a^2 + R^2 + 2aR\cos\Phi} = \sqrt{a^2 + {R^*}^2 + 2aR^*\cos\Phi}.$$

These two equations imply that $R^* = a^2/R$ and $q = -a/R$, and with these, $U + g$ vanishes identically on $r = a$. Thus, the Green's function for the Laplacian inside a sphere of radius a is

$$\begin{aligned} G(r,\phi,\theta;R,\Phi,\Theta) &= \frac{-1}{4\pi\sqrt{r^2 + R^2 - 2Rr[\cos\phi\cos\Phi + \sin\phi\sin\Phi\cos(\theta - \Theta)]}} \\ &\quad + \frac{a}{4\pi R\sqrt{r^2 + \left(\frac{a^2}{R}\right)^2 - 2r\left(\frac{a^2}{R}\right)[\cos\phi\cos\Phi + \sin\phi\sin\Phi\cos(\theta - \Theta)]}} \\ &= \frac{-1}{4\pi\sqrt{r^2 + R^2 - 2Rr[\cos\phi\cos\Phi + \sin\phi\sin\Phi\cos(\theta - \Theta)]}} \\ &\quad + \frac{a}{4\pi\sqrt{R^2r^2 + a^4 - 2a^2Rr[\cos\phi\cos\Phi + \sin\phi\sin\Phi\cos(\theta - \Theta)]}}. \end{aligned} \tag{13.38}$$

EXERCISES 13.2

1. Show that coefficients c_{mn} in representation 13.23 can be obtained by substituting $v = u_{mn}(x, y)$ and $u = G(x, y; X, Y)$ in Green's identity 13.14a.

2. Show that when $u_n(x, y)$ are orthonormal eigenfunctions of the eigenvalue problem

$$\nabla^2 u + \lambda^2 u = 0, \quad (x, y) \text{ in } A, \tag{13.39a}$$
$$u(x, y) = 0, \quad (x, y) \text{ on } \beta(A), \tag{13.39b}$$

associated with the Dirichlet problem

$$\nabla^2 u = F(x,y), \quad (x,y) \text{ in } A, \tag{13.40a}$$
$$u(x,y) = K(x,y), \quad (x,y) \text{ on } \beta(A), \tag{13.40b}$$

the full eigenfunction expansion for the Green's function is

$$G(x,y;X,Y) = \sum_{n=1}^{\infty} \frac{u_n(X,Y)u_n(x,y)}{-\lambda_n^2}. \tag{13.41}$$

(This expansion should be compared with that in Exercise 25 of Section 12.3 for the Green's function of an ODE.)

In Exercises 3–8 use Exercise 2 (and its extension to three dimensions) to find full eigenfunction expansions for the Green's function associated with the Dirichlet problem for Poisson's equation on the given domain.

3. $0 \le r < a$, $-\pi < \theta \le \pi$

4. $0 \le r < a$, $0 < \theta < \pi$

5. $0 \le r < a$, $0 < \theta < L$

6. $0 < x < L$, $0 < y < L'$, $0 < z < L''$

7. $0 \le r < a$, $-\pi < \theta \le \pi$, $0 < z < L$

8. $0 \le r < a$, $0 \le \phi \le \pi$, $-\pi < \theta \le \pi$

9. Use the method of images and the result of Example 13.4 to find the Green's function for the Dirichlet problem associated with Poisson's equation in a hemisphere of radius a.

10. Use a "modified" method of images to find the Green's function for the Dirichlet problem associated with the two-dimensional Laplacian on a circle of radius a. Assume that g consists of a potential due to an exterior, negative unit point source plus a constant potential.

11. Use the result of Exercise 10 and the method of images to find the Green's function for the Dirichlet problem associated with Poisson's equation on a semicircle $0 < r < a$, $0 < \theta < \pi$. How does it compare with the representation in Exercise 4.

12. Use the method of images to find the Green's function for the Dirichlet problem for the Laplacian on the rectangle $0 < x < L$, $0 < y < L'$.

13. In this exercise we use a partial eigenfunction expansion to find Green's function 13.36 for problem 13.33.

(a) Show that the partial eigenfunction expansion for $G(r,\theta;R,\Theta)$ is

$$G(r,\theta;R,\Theta) = \frac{A_0(r)}{\sqrt{2\pi}} + \sum_{n=1}^{\infty}\left[A_n(r)\frac{\cos n\theta}{\sqrt{\pi}} + B_n(r)\frac{\sin n\theta}{\sqrt{\pi}}\right].$$

(b) Substitute the expansion in part (a) into PDE 13.33a, and expand $\delta(r-R)\delta(\theta-\Theta)/r$ in a Fourier series to obtain the following boundary value problems for the coefficients:

$$\frac{d}{dr}\left(r\frac{dA_0}{dr}\right) = \frac{\delta(r-R)}{\sqrt{2\pi}}, \quad A_0(a) = 0;$$
$$\frac{d}{dr}\left(r\frac{dA_n}{dr}\right) - \frac{n^2}{r}A_n = \delta(r-R)\frac{\cos n\Theta}{\sqrt{\pi}}, \quad A_n(a) = 0;$$
$$\frac{d}{dr}\left(r\frac{dB_n}{dr}\right) - \frac{n^2}{r}B_n = \delta(r-R)\frac{\sin n\Theta}{\sqrt{\pi}}, \quad B_n(a) = 0.$$

(c) The systems in part (b) are "singular" in the sense that there is only one boundary condition and the coefficient r in the derivative term vanishes at $r = 0$. As a result, equations 12.33 and 12.34 in Section 12.3 cannot be used to find A_n and B_n. Instead, use properties 12.26a–c from Section 12.3 and the one boundary condition to show that

$$A_0(r) = \begin{cases} \dfrac{\ln(R/a)}{\sqrt{2\pi}}, & 0 \le r \le R \\ \dfrac{\ln(r/a)}{\sqrt{2\pi}}, & R < r \le a, \end{cases}$$

$$A_n(r) = \begin{cases} \dfrac{\cos n\Theta}{2\sqrt{\pi}n}\left[\left(\dfrac{rR}{a^2}\right)^n - \left(\dfrac{r}{R}\right)^n\right], & 0 \le r \le R \\ \dfrac{\cos n\Theta}{2\sqrt{\pi}n}\left[\left(\dfrac{rR}{a^2}\right)^n - \left(\dfrac{R}{r}\right)^n\right], & R < r \le a, \end{cases}$$

$$B_n(r) = \begin{cases} \dfrac{\sin n\Theta}{2\sqrt{\pi}n}\left[\left(\dfrac{rR}{a^2}\right)^n - \left(\dfrac{r}{R}\right)^n\right], & 0 \le r \le R \\ \dfrac{\sin n\Theta}{2\sqrt{\pi}n}\left[\left(\dfrac{rR}{a^2}\right)^n - \left(\dfrac{R}{r}\right)^n\right], & R < r \le a. \end{cases}$$

(d) Find $G(r,\theta;R,\Theta)$ and use identity 13.35 to reduce the function to the form in equation 13.36.

14. Use the technique of Exercise 13 to find a partial eigenfunction expansion for the Green's function of the Dirichlet problem for the Laplacian on the semicircle $0 < r < a$, $0 < \theta < \pi$. Show that it can be expressed in the form of Exercise 11.

15. Use the technique of Exercise 13 to find the partial eigenfunction expansion for the Green's function of Exercise 5.

16. Find a partial eigenfunction expansion for the Green's function of Exercise 6 using eigenfunctions in x and y.

17. Show that when $u_n(x,y)$ are orthonormal eigenfunctions of eigenvalue problem 13.21, the full eigenfunction expansion for the Green's function of the boundary value problem

$$\nabla^2 u + k^2 u = F(x,y), \quad (x,y) \text{ in } A, \tag{13.42a}$$

$$u(x,y) = K(x,y), \quad (x,y) \text{ on } \beta(A), \tag{13.42b}$$

is

$$G(x,y;X,Y) = \sum_{n=1}^{\infty} \frac{u_n(X,Y)u_n(x,y)}{k^2 - \lambda_n^2}, \tag{13.43}$$

provided $k \ne \lambda_n$ for any n. (The exceptional case is discussed in Exercise 8 of Section 13.3.)

In Exercises 18–24 use Exercise 17 to state Green's functions for problem 13.42 on the given domain. (See Example 13.1 and Exercises 3–8 for eigenpairs.)

18. $0 < x < L$, $0 < y < L'$

19. $0 \le r < a$, $-\pi < \theta \le \pi$

20. $0 \le r < a$, $0 < \theta < \pi$

21. $0 < r < a$, $0 < \theta < L$

22. $0 < x < L$, $0 < y < L'$, $0 < z < L''$

23. $0 \le r < a$, $-\pi < \theta \le \pi$, $0 < z < L$

24. $0 \le r < a$, $0 \le \phi \le \pi$, $-\pi < \theta \le \pi$

25. Derive the free-space Green's function for the 3-dimensional Laplacian by taking the source at the origin and using spherical coordinates centred there.

26. In this exercise we give a derivation of free-space Green's functions for the two-dimensional Laplacian that can be used to find free-space Green's functions for Helmholtz operators.

(a) Show that $G(r;0) = C\ln r + D$ is a general solution of the homogeneous version of equation 13.32.

(b) By substituting $v = G(r;0)$ and $u = 1$ in Green's second identity 13.14a where A is a circle of radius ϵ centred at the source $r = 0$, show that

$$\epsilon\frac{\partial G(\epsilon;0)}{\partial r} = \frac{1}{2\pi}.$$

(c) Reason that $G(r;0)$ must satisfy

$$r\frac{\partial G(r;0)}{\partial r} = \frac{1}{2\pi} \qquad \text{and} \qquad \lim_{r\to 0} r\frac{\partial G(r;0)}{\partial r} = \frac{1}{2\pi}.$$

(d) Use the results of parts (a) and (c), to find $G(r;0)$.

27. Use the technique of Exercise 26 to derive the free-space Green's function in Table 13.1 for the three-dimensional Laplacian.

28. Use the technique of Exercise 26 to derive the free-space Green's functions in Table 13.1 for the three-dimensional Helmholtz operator $\nabla^2 + k^2$. Hint: Set $G(r;0) = H(r)/r$ in the homogeneous differential equation for $G(r;0)$.

29. Use the technique of Exercise 26 to derive the free-space Green's function in Table 13.1 for the two-dimensional Helmholtz operator $\nabla^2 + k^2$.