§10.4 Applications to Partial Differential Equations on Bounded Domains

Laplace transforms can be used to eliminate the time variable from initial boundary value problems. This reduces the PDE to an ODE or a PDE with one fewer variable. We illustrate with the following examples.

Example 10.13 Solve the heat conduction problem

$$\frac{\partial U}{\partial t} = k\frac{\partial^2 U}{\partial x^2}, \quad 0 < x < L, \quad t > 0, \tag{10.26a}$$

$$U(0,t) = 0, \quad t > 0, \tag{10.26b}$$

$$U(L,t) = 0, \quad t > 0, \tag{10.26c}$$

$$U(x,0) = x, \quad 0 < x < L. \tag{10.26d}$$

Solution When we take Laplace transforms with respect to t on both sides of the PDE and use property 10.7a,

$$s\tilde{U}(x,s) - x = k\frac{\partial^2 \tilde{U}}{\partial x^2}.$$

Thus, $\tilde{U}(x,s)$ must satisfy the ODE

$$\frac{d^2\tilde{U}}{dx^2} - \frac{s}{k}\tilde{U} = -\frac{x}{k} \tag{10.27a}$$

subject to the transforms of boundary conditions 10.26b,c,

$$\tilde{U}(0,s) = 0, \tag{10.27b}$$

$$\tilde{U}(L,s) = 0. \tag{10.27c}$$

A general solution of the ODE is

$$\tilde{U}(x,s) = C_1 \cosh\sqrt{\frac{s}{k}}x + C_2 \sinh\sqrt{\frac{s}{k}}x + \frac{x}{s},$$

and the boundary conditions require

$$0 = C_1, \qquad 0 = C_1 \cosh\sqrt{\frac{s}{k}}L + C_2 \sinh\sqrt{\frac{s}{k}}L + \frac{L}{s}.$$

From these,

$$\tilde{U}(x,s) = \frac{1}{s}\left(x - \frac{L\sinh\sqrt{s/k}x}{\sinh\sqrt{s/k}L}\right). \tag{10.28}$$

It remains now to find the inverse transform of $\tilde{U}(x,s)$. We do this by calculating residues of $e^{st}\tilde{U}(x,s)$ at the singularities of $\tilde{U}(x,s)$. To discover the nature of the singularity at $s = 0$, we expand $\tilde{U}(x,s)$ in a Laurent series around $s = 0$,

$$\begin{aligned}\tilde{U}(x,s) &= \frac{1}{s}\left\{x - \frac{L[\sqrt{s/k}x + (\sqrt{s/k}x)^3/3! + \cdots]}{\sqrt{s/k}L + (\sqrt{s/k}L)^3/3! + \cdots}\right\}\\ &= \frac{1}{s}\left[x - \frac{x + sx^3/(6k) + \cdots}{1 + sL^2/(6k) + \cdots}\right]\\ &= \frac{1}{s}\left[\frac{sx(L^2 - x^2)}{6k} + \cdots\right] = \frac{x(L^2 - x^2)}{6k} + \text{terms in } s, s^2, \ldots.\end{aligned}$$

It follows that $\tilde{U}(x,s)$ has a removable singularity at $s=0$.

The remaining singularities of $\tilde{U}(x,s)$ occur at the zeros of $\sinh\sqrt{s/k}L$; that is, when $\sqrt{s/k}L = n\pi i$ or $s=-n^2\pi^2k/L^2$, n a positive integer. Because the derivative of $\sinh\sqrt{s/k}L$ does not vanish at $s=-n^2\pi^2k/L^2$, this function has zeros of order 1 at $s=-n^2\pi^2k/L^2$. It follows that $\tilde{U}(x,s)$ has poles of order 1 at these singularities, and, according to formula 10.25, the residue of $e^{st}\tilde{U}(x,s)$ at $s=-n^2\pi^2k/L^2$ is

$$\begin{aligned}\operatorname{Res}\left[e^{st}\tilde{U}(x,s), -\frac{n^2\pi^2k}{L^2}\right] &= \lim_{s\to -n^2\pi^2k/L^2}\left(s+\frac{n^2\pi^2k}{L^2}\right)\frac{e^{st}}{s}\left(x-\frac{L\sinh\sqrt{s/k}x}{\sinh\sqrt{s/k}L}\right)\\ &= -\frac{e^{-n^2\pi^2kt/L^2}}{-n^2\pi^2k/L^2}L\sinh\frac{n\pi xi}{L}\lim_{s\to -n^2\pi^2k/L^2}\left(\frac{s+n^2\pi^2k/L^2}{\sinh\sqrt{s/k}L}\right).\end{aligned}$$

L'Hôpital's rule yields

$$\begin{aligned}\operatorname{Res}\left[e^{st}\tilde{U}(x,s), -\frac{n^2\pi^2k}{L^2}\right] &= \frac{iL^3}{n^2\pi^2k}e^{-n^2\pi^2kt/L^2}\sin\frac{n\pi x}{L}\lim_{s\to -n^2\pi^2k/L^2}\frac{1}{\dfrac{L}{2\sqrt{ks}}\cosh\sqrt{s/k}L}\\ &= \frac{2iL^2}{n^2\pi^2k}e^{-n^2\pi^2kt/L^2}\sin\frac{n\pi x}{L}\frac{1}{\dfrac{L}{n\pi ki}\cosh n\pi i}\\ &= \frac{2L}{n\pi}(-1)^{n+1}e^{-n^2\pi^2kt/L^2}\sin\frac{n\pi x}{L}.\end{aligned}$$

We sum these residues to find the inverse Laplace transform of $\tilde{U}(x,s)$,

$$U(x,t) = \frac{2L}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}e^{-n^2\pi^2kt/L^2}\sin\frac{n\pi x}{L}.\bullet \tag{10.29}$$

Before proceeding to further problems, some general comments are appropriate:

1. In the above example, the Laplace transform was applied to the time variable to eliminate the time derivative from the PDE and obtain an ODE in $\tilde{U}(x,s)$. The Laplace transform cannot be applied to the space variable x, because the range of x is only $0\le x\le L$. It is the power of finite Fourier transforms to eliminate the space variable, not the Laplace transform.
2. The Laplace transform immediately incorporates the initial condition into the solution, and boundary conditions on $U(x,t)$ become boundary conditions for $\tilde{U}(x,s)$. Contrast this with finite Fourier transforms, which immediately incorporate boundary conditions and use the initial condition on $U(x,t)$ as an initial condition for $\tilde{U}(\lambda_n,t)$.
3. Mathematically, the solution is not complete because the existence of a sequence of contours satisfying the properties of Theorem 10.6 has not been established, but we omit this part of the problem. We could circumvent this difficulty by now verifying that function 10.29 does indeed satisfy initial boundary value problem 10.26.

Problems with arbitrary initial conditions are more difficult to handle. This is illustrated in the next example.

Example 10.14 Solve the vibration problem

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}, \quad 0 < x < L, \quad t > 0, \tag{10.30a}$$

$$y(0,t) = 0, \quad t > 0, \tag{10.30b}$$

$$y(L,t) = 0, \quad t > 0, \tag{10.30c}$$

$$y(x,0) = f(x), \quad 0 < x < L, \tag{10.30d}$$

$$y_t(x,0) = 0, \quad 0 < x < L, \tag{10.30e}$$

(see Exercise 20 in Section 4.2, with $g(x) \equiv 0$).

Solution When we take Laplace transforms of the PDE with respect to t and use initial conditions 10.30d,e in property 10.7b,

$$s^2 \tilde{y} - s f(x) = c^2 \frac{\partial^2 \tilde{y}}{\partial x^2}.$$

Thus, $\tilde{y}(x,s)$ must satisfy the ODE

$$\frac{d^2 \tilde{y}}{dx^2} - \frac{s^2}{c^2} \tilde{y} = -\frac{s}{c^2} f(x) \tag{10.31a}$$

subject to transforms of boundary conditions 10.30b,c,

$$\tilde{y}(0,s) = 0, \quad \tilde{y}(L,s) = 0. \tag{10.31b}$$

Variation of parameters (see Section 4.3) leads to the following form for a general solution of ODE 10.31a

$$\tilde{y}(x,s) = C_1 \cosh \frac{sx}{c} + C_2 \sinh \frac{sx}{c} - \frac{1}{c} \int_0^x f(u) \sinh \frac{s}{c}(x-u)\, du.$$

Boundary conditions 10.31b require

$$0 = C_1, \qquad 0 = C_1 \cosh \frac{sL}{c} + C_2 \sinh \frac{sL}{c} - \frac{1}{c} \int_0^L f(u) \sinh \frac{s}{c}(L-u)\, du,$$

from which

$$\begin{aligned} \tilde{y}(x,s) &= \frac{\sinh \frac{sx}{c}}{c \sinh \frac{sL}{c}} \int_0^L f(u) \sinh \frac{s}{c}(L-u)\, du - \frac{1}{c} \int_0^x f(u) \sinh \frac{s}{c}(x-u)\, du \\ &= \int_0^L f(u) \tilde{p}(x,u,s)\, du - \frac{1}{c} \int_0^x f(u) \sinh \frac{s}{c}(x-u)\, du, \end{aligned} \tag{10.32a}$$

where

$$\tilde{p}(x,u,s) = \frac{\sinh \frac{sx}{c} \sinh \frac{s}{c}(L-u)}{c \sinh \frac{sL}{c}}. \tag{10.32b}$$

To obtain $y(x,t)$ by residues requires the singularities of $\tilde{y}(x,s)$. Provided $f(x)$ is piecewise continuous, integration with respect to u in 10.32a and any differentiation with respect to s can be interchanged, and therefore the second integral in

10.32a has no singularities. Singularities of the first integral are determined by those of $\tilde{p}(x,u,s)$. For the singularity at $s=0$, we note that

$$\begin{aligned}
\tilde{p}(x,u,s) &= \frac{1}{c}\sinh\frac{s}{c}(L-u)\left(\frac{\sinh\dfrac{sx}{c}}{\sinh\dfrac{sL}{c}}\right)\\
&= \frac{1}{c}\left[\frac{s}{c}(L-u)+\frac{s^3}{3!c^3}(L-u)^3+\cdots\right]\left[\frac{\dfrac{sx}{c}+\dfrac{1}{3!}\left(\dfrac{sx}{c}\right)^3+\cdots}{\dfrac{sL}{c}+\dfrac{1}{3!}\left(\dfrac{sL}{c}\right)^3+\cdots}\right]\\
&= \left[\frac{s}{c^2}(L-u)+\frac{s^3}{6c^4}(L-u)^3+\cdots\right]\left(\frac{x+\dfrac{x^3s^2}{6c^2}+\cdots}{L+\dfrac{L^3s^2}{6c^2}+\cdots}\right)\\
&= \frac{s}{c^2}(L-u)\frac{x}{L}+\text{terms in } s^2, s^3, \ldots,
\end{aligned}$$

and therefore $\tilde{p}(x,u,s)$ has a removable singularity at $s=0$. The remaining singularities of $\tilde{p}(x,u,s)$ are $s=n\pi ci/L$, n a nonzero integer. Because the derivative of $\sinh(sL/c)$ does not vanish at $s=n\pi ci/L$, these singularities are poles of order 1. According to formula 10.25, the residue of $\tilde{p}(x,u,s)$ at $s=n\pi ci/L$ is

$$\begin{aligned}
\operatorname{Res}\left[\tilde{p}(x,u,s),\frac{n\pi ci}{L}\right] &= \lim_{s\to n\pi ci/L}\left(s-\frac{n\pi ci}{L}\right)\tilde{p}(x,u,s)\\
&= \lim_{s\to n\pi ci/L}\left(s-\frac{n\pi ci}{L}\right)\frac{\sinh\dfrac{sx}{c}\sinh\dfrac{s}{c}(L-u)}{c\sinh\dfrac{sL}{c}}\\
&= \sinh\frac{n\pi xi}{L}\sinh\frac{n\pi i(L-u)}{L}\lim_{s\to n\pi ci/L}\frac{s-\dfrac{n\pi ci}{L}}{c\sinh\dfrac{sL}{c}}\\
&= -\sin\frac{n\pi x}{L}\sin\frac{n\pi}{L}(L-u)\lim_{s\to n\pi ci/L}\frac{1}{L\cosh\dfrac{sL}{c}} \qquad \text{(by l'Hôpital's rule)}\\
&= \frac{(-1)^n}{L}\sin\frac{n\pi x}{L}\sin\frac{n\pi u}{L}\frac{1}{\cosh n\pi i}\\
&= \frac{1}{L}\sin\frac{n\pi x}{L}\sin\frac{n\pi u}{L}.
\end{aligned}$$

The residue of e^{st} times the first integral in 10.32a at $s=n\pi ci/L$ is now

$$\lim_{s\to n\pi ci/L}\left(s-\frac{n\pi ci}{L}\right)e^{st}\int_0^L f(u)\tilde{p}(x,u,s)\,du.$$

When we interchange the limit on s with the integration with respect to u, the residue becomes

$$\begin{aligned}
\int_0^L \lim_{s\to n\pi ci/L}\left[e^{st}\left(s-\frac{n\pi ci}{L}\right)f(u)\tilde{p}(x,u,s)\right]du &= \int_0^L e^{n\pi cti/L}f(u)\frac{1}{L}\sin\frac{n\pi x}{L}\sin\frac{n\pi u}{L}du\\
&= \frac{1}{L}e^{n\pi cti/L}\sin\frac{n\pi x}{L}\int_0^L f(u)\sin\frac{n\pi u}{L}du.
\end{aligned}$$

The inverse transform of $\tilde{y}(x,s)$ is the sum of all such residues,

$$y(x,t) = \frac{1}{L} \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} e^{n\pi cti/L} \sin\frac{n\pi x}{L} \int_0^L f(u) \sin\frac{n\pi u}{L} du. \tag{10.33}$$

To simplify this summation, we divide it into two parts,

$$\begin{aligned} y(x,t) = \frac{1}{L} \sum_{n=1}^{\infty} e^{n\pi cti/L} \sin\frac{n\pi x}{L} \int_0^L f(u)\sin\frac{n\pi u}{L} du \\ + \frac{1}{L} \sum_{n=-\infty}^{-1} e^{n\pi cti/L} \sin\frac{n\pi x}{L} \int_0^L f(u)\sin\frac{n\pi u}{L} du, \end{aligned}$$

and replace n by $-n$ in the second summation,

$$\begin{aligned} y(x,t) &= \frac{1}{L} \sum_{n=1}^{\infty} e^{n\pi cti/L} \sin\frac{n\pi x}{L} \int_0^L f(u)\sin\frac{n\pi u}{L} du \\ &\quad + \frac{1}{L} \sum_{n=1}^{\infty} e^{-n\pi cti/L} \sin\left(\frac{-n\pi x}{L}\right) \int_0^L f(u)\sin\left(\frac{-n\pi u}{L}\right) du \\ &= \frac{1}{L} \sum_{n=1}^{\infty} \sin\frac{n\pi x}{L} (e^{n\pi cti/L} + e^{-n\pi cti/L}) \int_0^L f(u)\sin\frac{n\pi u}{L} du \\ &= \sum_{n=1}^{\infty} a_n \cos\frac{n\pi ct}{L} \sin\frac{n\pi x}{L} \quad \text{where } a_n = \frac{2}{L}\int_0^L f(u)\sin\frac{n\pi u}{L} du. \end{aligned} \tag{10.34}$$

This is identical to the solution obtained by separation of variables in Exercise 20 in Section 4.2 when $g(x)$ is set equal to zero.●

Examples 10.13 and 10.14 were homogeneous problems. Convolutions can be used to handle problems with unspecified nonhomogeneities.

Example 10.15 Solve Example 10.13 if the end $x = 0$ of the rod has a prescribed temperature $f(t)$ and the initial temperature is zero throughout. Compare the solution with that obtained by variation of constants and by finite Fourier transforms.

Solution The initial boundary value problem is

$$\frac{\partial U}{\partial t} = k\frac{\partial^2 U}{\partial x^2}, \quad 0 < x < L, \quad t > 0, \tag{10.35a}$$

$$U(0,t) = f(t), \quad t > 0, \tag{10.35b}$$

$$U(L,t) = 0, \quad t > 0, \tag{10.35c}$$

$$U(x,0) = 0, \quad 0 < x < L. \tag{10.35d}$$

When the Laplace transform is applied to the PDE and initial temperature 10.35d is used, the transform $\tilde{U}(x,s)$ must satisfy the ODE

$$\frac{d^2\tilde{U}}{dx^2} - \frac{s}{k}\tilde{U} = 0, \tag{10.36a}$$

$$\tilde{U}(0,s) = \tilde{f}(s), \tag{10.36b}$$

$$\tilde{U}(L,s) = 0. \tag{10.36c}$$

The solution of this system is

$$\tilde{U}(x,s) = \frac{\tilde{f}(s)\sinh\sqrt{s/k}(L-x)}{\sinh\sqrt{s/k}L}. \tag{10.37}$$

To find the inverse transform of this function, first consider finding the inverse of $\tilde{p}(x,s) = \sinh\sqrt{s/k}(L-x)/\sinh\sqrt{s/k}L$. This function has singularities when $\sqrt{s/k}L = n\pi i$ or $s = -n^2\pi^2 k/L^2$, n a nonnegative integer. Expansion of $\tilde{p}(x,s)$ in a Laurent series around $s = 0$ immediately shows that $s = 0$ is a removable singularity. The remaining singularities are poles of order 1, and the residue of $e^{st}\tilde{p}(x,s)$ at $s = -n^2\pi^2 k/L^2$ is

$$\begin{aligned}
\operatorname{Res}\left[e^{st}\tilde{p}(x,s), -\frac{n^2\pi^2 k}{L^2}\right] &= \lim_{s\to -n^2\pi^2 k/L^2}\left(s + \frac{n^2\pi^2 k}{L^2}\right)e^{st}\frac{\sinh\sqrt{s/k}(L-x)}{\sinh\sqrt{s/k}L} \\
&= e^{-n^2\pi^2 kt/L^2}\sinh\frac{n\pi i(L-x)}{L}\lim_{s\to -n^2\pi^2 k/L^2}\left(\frac{s + n^2\pi^2 k/L^2}{\sinh\sqrt{s/k}L}\right) \\
&= ie^{-n^2\pi^2 kt/L^2}\sin\frac{n\pi(L-x)}{L}\lim_{s\to -n^2\pi^2 k/L^2}\frac{1}{\dfrac{L}{2\sqrt{ks}}\cosh\sqrt{s/k}L} \\
&= ie^{-n^2\pi^2 kt/L^2}(-1)^{n+1}\sin\frac{n\pi x}{L}\frac{2nk\pi i}{L^2\cosh n\pi i} \\
&= \frac{2nk\pi}{L^2}e^{-n^2\pi^2 kt/L^2}\sin\frac{n\pi x}{L}.
\end{aligned}$$

Convolutions can now be used to invert $\tilde{U}(x,s)$,

$$\begin{aligned}
U(x,t) = \mathcal{L}^{-1}\{\tilde{f}(s)\tilde{p}(x,s)\} &= \int_0^t f(u)p(x,t-u)\,du \\
&= \int_0^t f(u)\left[\frac{2k\pi}{L^2}\sum_{n=1}^{\infty} ne^{-n^2\pi^2 k(t-u)/L^2}\sin\frac{n\pi x}{L}\right]du \\
&= \frac{2k\pi}{L^2}\sum_{n=1}^{\infty} c_n(t)\sin\frac{n\pi{`}x}{L},
\end{aligned} \tag{10.38a}$$

where

$$c_n(t) = n\int_0^t f(u)e^{-n^2\pi^2 k(t-u)/L^2}du. \tag{10.38b}$$

With variation of constants (see Section 4.3), the dependent variable is changed to $V(x,t) = U(x,t) - f(t)(1 - x/L)$, resulting in a problem with homogeneous boundary conditions for $V(x,t)$,

$$\begin{aligned}
\frac{\partial V}{\partial t} &= k\frac{\partial^2 V}{\partial x^2} - f'(t)\left(1 - \frac{x}{L}\right), \quad 0 < x < L, \quad t > 0, \\
V(0,t) &= 0, \quad t > 0, \\
V(L,t) &= 0, \quad t > 0, \\
V(x,0) &= -f(0)\left(1 - \frac{x}{L}\right) = 0, \quad 0 < x < L,
\end{aligned}$$

provided we assume that $f(0) = 0$. (The $f(0) \neq 0$ situation is discussed in Exercise 12.) Variation of constants

$$V(x,t) = \sum_{n=1}^{\infty} a_n(t) \sin \frac{n\pi x}{L}$$

leads to

$$a_n(t) = \frac{-2}{n\pi} \int_0^t f'(u) e^{-n^2\pi^2 k(t-u)/L^2} du,$$

and therefore

$$U(x,t) = f(t)\left(1 - \frac{x}{L}\right) - \frac{2}{\pi} \sum_{n=1}^{\infty} \left[\frac{1}{n} \int_0^t f'(u) e^{-n^2\pi^2 k(t-u)/L^2} du\right] \sin \frac{n\pi x}{L}. \quad (10.39)$$

That this is identical to solution 10.38 is verified by integrating expression 10.38b by parts,

$$\begin{aligned} c_n(t) &= n\left\{\frac{L^2}{n^2\pi^2 k} f(u) e^{-n^2\pi^2 k(t-u)/L^2}\right\}_0^t - n \int_0^t \frac{L^2}{n^2\pi^2 k} f'(u) e^{-n^2\pi^2 k(t-u)/L^2} du \\ &= \frac{L^2}{n\pi^2 k} f(t) - \frac{L^2}{n\pi^2 k} \int_0^t f'(u) e^{-n^2\pi^2 k(t-u)/L^2} du, \end{aligned}$$

and substituting into equation 10.38a,

$$\begin{aligned} U(x,t) &= \frac{2k\pi}{L^2} \sum_{n=1}^{\infty} \left[\frac{L^2}{nk\pi^2} f(t) - \frac{L^2}{nk\pi^2} \int_0^t f'(u) e^{-n^2\pi^2 k(t-u)/L^2} du\right] \sin \frac{n\pi x}{L} \\ &= f(t) \sum_{n=1}^{\infty} \frac{2}{n\pi} \sin \frac{n\pi x}{L} - \frac{2}{\pi} \sum_{n=1}^{\infty} \left[\frac{1}{n} \int_0^t f'(u) e^{-n^2\pi^2 k(t-u)/L^2} du\right] \sin \frac{n\pi x}{L}. \end{aligned}$$

This is identical to solution 10.39 when we notice that the coefficients in the Fourier sine series of $1 - x/L$ are $2/(n\pi)$.

The finite Fourier transform

$$\tilde{f}(\lambda_n) = \int_0^L f(x) \sqrt{\frac{2}{L}} \sin \frac{n\pi x}{L} dx$$

applied to problem 10.35 gives the solution in form 10.38.•

When we write solution 10.38 for problem 10.35 in the form

$$U(x,t) = \frac{2k\pi}{L^2} \sum_{n=1}^{\infty} b_n e^{-n^2\pi^2 kt/L^2} \sin \frac{n\pi x}{L} \quad \text{where } b_n = n \int_0^t f(u) e^{n^2\pi^2 ku/L^2} du, \quad (10.40)$$

we see that the exponentials in the series enhance convergence for large values of t. For instance, if the temperature of the left end is maintained at 100°C for $t > 0$, the temperature function reduces to

$$U(x,t) = \frac{200}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} (1 - e^{-n^2\pi^2 kt/L^2}) \sin \frac{n\pi x}{L}, \quad (10.41)$$

which can also be expressed in the form

$$U(x,t) = 100\left(1 - \frac{x}{L}\right) - \frac{200}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}e^{-n^2\pi^2 kt/L^2}\sin\frac{n\pi x}{L}. \tag{10.42}$$

Suppose the rod is 1/5 m in length and is made from stainless steel with thermal diffusivity $k = 3.87 \times 10^{-6}$ m^2/s. Consider finding the temperature at the midpoint $x = 1/10$ of the rod at the four times $t = 2, 5, 30$ and 100 minutes. Series 10.42 gives

$$U(0.1,120) = 100\left(1 - \frac{1}{2}\right) - \frac{200}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}e^{-0.1145861n^2}\sin\frac{n\pi}{2} = 0.10°\text{C},$$

$$U(0.1,300) = 100\left(1 - \frac{1}{2}\right) - \frac{200}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}e^{-0.28646526n^2}\sin\frac{n\pi}{2} = 3.80°\text{C},$$

$$U(0.1,1800) = 100\left(1 - \frac{1}{2}\right) - \frac{200}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}e^{-1.7187915n^2}\sin\frac{n\pi}{2} = 38.6°\text{C},$$

$$U(0.1,6000) = 100\left(1 - \frac{1}{2}\right) - \frac{200}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}e^{-5.7293052n^2}\sin\frac{n\pi}{2} = 49.8°\text{C}.$$

To obtain these temperatures, we required only four nonzero terms from the first series, three from the second, one each from the third and fourth. This substantiates our claim that as t increases, fewer and fewer terms in series 10.42 are required for accurate calculations of temperature.

Laplace transforms can be used to give a completely different representation for the temperature in the rod when $f(t) = 100$. To find this representation, we return to expression 10.37 for the Laplace transform $\tilde{U}(x,s)$ of $U(x,t)$ and set $\tilde{f}(s) = 100/s$, the transform of $f(t) = 100$,

$$\begin{aligned}\tilde{U}(x,s) &= \frac{100\sinh\sqrt{s/k}(L-x)}{s\,\sinh\sqrt{s/k}L} = \frac{100}{s}\frac{e^{\sqrt{s/k}(L-x)} - e^{-\sqrt{s/k}(L-x)}}{e^{\sqrt{s/k}L} - e^{-\sqrt{s/k}L}} \\ &= \frac{100}{s}\frac{e^{-\sqrt{s/k}L}[e^{\sqrt{s/k}(L-x)} - e^{-\sqrt{s/k}(L-x)}]}{1 - e^{-2\sqrt{s/k}L}}.\end{aligned}$$

If we regard $1/(1-e^{-2\sqrt{s/k}L})$ as the sum of an infinite geometric series with common ratio $e^{-2\sqrt{s/k}L}$, we may write

$$\begin{aligned}\tilde{U}(x,s) &= \frac{100}{s}[e^{-\sqrt{s/k}x} - e^{-\sqrt{s/k}(2L-x)}]\sum_{n=0}^{\infty}e^{-2n\sqrt{s/k}L} \\ &= 100\sum_{n=0}^{\infty}\left[\frac{e^{-\sqrt{s/k}(2nL+x)}}{s} - \frac{e^{-\sqrt{s/k}[2(n+1)L-x]}}{s}\right].\end{aligned} \tag{10.43}$$

Tables of Laplace transforms indicate that

$$\mathcal{L}^{-1}\left\{\frac{e^{-a\sqrt{s}}}{s}\right\} = \text{erfc}\left(\frac{a}{2\sqrt{t}}\right),$$

where erfc(x) is the complementary error function in equation 10.16. Hence, $U(x,t)$ may be expressed as a series of complementary error functions,

$$U(x,t) = 100\sum_{n=0}^{\infty}\left[\operatorname{erfc}\left(\frac{2nL+x}{2\sqrt{kt}}\right) - \operatorname{erfc}\left(\frac{2(n+1)L-x}{2\sqrt{kt}}\right)\right]$$
$$= 100\sum_{n=0}^{\infty}\left[\operatorname{erf}\left(\frac{2(n+1)L-x}{2\sqrt{kt}}\right) - \operatorname{erf}\left(\frac{2nL+x}{2\sqrt{kt}}\right)\right], \qquad (10.44)$$

where we have used the fact that erfc(x) $= 1 -$ erf(x). The error function erf(x) is defined as

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-u^2}\,du. \qquad (10.45)$$

This representation of $U(x,t)$ is valuable for small values of t (as opposed to 10.42, which converges rapidly for large t). To see this, consider temperature at the mid-point of the above stainless steel rod at $t = 300$ s,

$$U(0.1,300) = 100\sum_{n=0}^{\infty}\left[\operatorname{erf}\left(\frac{2(n+1)/5-0.1}{2\sqrt{3.87\times 10^{-6}(300)}}\right) - \operatorname{erf}\left(\frac{2n/5+0.1}{2\sqrt{3.87\times 10^{-6}(300)}}\right)\right].$$

For $n > 0$, all terms in this series essentially vanish, and

$$U(0.1,300) = 100[\operatorname{erf}(4.40) - \operatorname{erf}(1.467)] = 3.80^\circ\text{C}.$$

For $t = 1800$,

$$U(0.1,1800) = 100\sum_{n=0}^{\infty}\left[\operatorname{erf}\left(\frac{2(n+1)/5-0.1}{2\sqrt{3.87\times 10^{-6}(1800)}}\right) - \operatorname{erf}\left(\frac{2n/5+0.1}{2\sqrt{3.87\times 10^{-6}(1800)}}\right)\right].$$

Once again, only the $n = 0$ term is required; it yields $U(0.1,1800) = 38.6^\circ$C. Finally, for $t = 6000$,

$$U(0.1,6000) = 100\sum_{n=0}^{\infty}\left[\operatorname{erf}\left(\frac{2(n+1)/5-0.1}{2\sqrt{3.87\times 10^{-6}(6000)}}\right) - \operatorname{erf}\left(\frac{2n/5+0.1}{2\sqrt{3.87\times 10^{-6}(6000)}}\right)\right].$$

In this case, the $n = 0$ and $n = 1$ terms give $U(0.1,6000) = 49.9^\circ$C. For larger values of t, more and more terms are required.

The error function representiation in equation 10.44 once again substantiates our claim in Section 6.6 that heat propagates with infinite speed. Because the error function is an increasing function of its argument, and the argument $(2nL+2L-x)/(2\sqrt{kt})$ of the first error function in 10.44 is greater than the second argument, $(2nL+x)/(2\sqrt{kt})$, it follows that each term in 10.44 is positive, Since this is true for every x in $0 < x < L$ and every $t > 0$, the temperature at every point in the rod for every $t > 0$ is positive. This means that the effect of changing the temperature of the end $x = 0$ of the rod from 0°C to 100°C at time $t = 0$ is instantaneously felt at every point in the rod. The amount of heat transmitted to other parts of the rod may be minute, but nonetheless, heat is transmitted instantaneously to all parts of the rod.

EXERCISES 10.4

Use Laplace transforms to solve all problems in this set of exercises.
Part A Heat Conduction

1. A homogeneous, isotropic rod with insulated sides has temperature $\sin m\pi x/L$ (m an integer) at time $t = 0$. For time $t > 0$, its ends ($x = 0$ and $x = L$) are held at temperature 0°C. Find a formula for temperature $U(x,t)$ in the rod for $0 < x < L$ and $t > 0$.

2. Solve Example 4.2 in Section 4.2 when the initial temperature is $U_0 =$ constant.

3. Repeat Exercise 1 if the initial temperature is 10°C throughout.

4. Solve Exercise 8 in Section 4.3.

5. Solve Exercise 2 in Section 4.2.

6. Solve Example 4.2 in Section 4.2 when the initial temperature is $f(x)$ (in place of x).

7. A homogeneous, isotropic rod with insulated sides is initially ($t = 0$) at temperature 0°C throughout. For $t > 0$, its left end, $x = 0$, is kept at 0°C and its right end, $x = L$, is kept at constant temperature U_L°C. Find two expressions for temperature in the rod, one in terms of exponentials in time and the other in terms of error functions.

8. A homogeneous, isotropic rod with insulated sides is initially ($t = 0$) at constant temperature U_0°C throughout. For $t > 0$, its end $x = 0$ is insulated, and heat is added to the end $x = L$ at a constant rate Q W/m^2. Find the temperature in the rod for $0 < x < L$ and $t > 0$.

9. (a) A homogeneous, isotropic rod with insulated sides has, for time $t > 0$, its ends at $x = 0$ and $x = L$ kept at temperature zero. Initially its temperature is Ax, where A is constant. Show that temperature in the rod can be expressed in two ways:

$$U(x,t) = \frac{2AL}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}e^{-n^2\pi^2kt/L^2}\sin\frac{n\pi x}{L},$$

and $$U(x,t) = A\left\{x - L\sum_{n=0}^{\infty}\left[\operatorname{erf}\left(\frac{(2n+1)L+x}{2\sqrt{kt}}\right) - \operatorname{erf}\left(\frac{(2n+1)L-x}{2\sqrt{kt}}\right)\right]\right\}.$$

(b) Which of the two solutions do you expect to converge more rapidly for small t? For large t?

(c) Verify your conjecture in part (b) by calculating the temperature at the midpoint of a stainless steel rod ($k = 3.87 \times 10^{-6}$) of length 1/5 m when $A = 500$ and (i) $t = 30$ s (ii) $t = 5$ min (iii) $t = 100$ min.

10. A homogeneous, isotropic rod with insulated sides is initially ($t = 0$) at temperature 0°C throughout. For $t > 0$, its left end, $x = 0$, is kept at 0°C and heat is added to the end $x = L$ at a constant rate $Q > 0$ W/m^2. Find two series representations for $U(x,t)$, one in terms of error functions and one in terms of time exponentials.

11. Solve Exercise 13 in Section 7.2.

12. Show that the Laplace transform solution and the eigenfunction expansion solution to the problem in Example 10.15 are identical when $f(0) \neq 0$.

13. A homogeneous, isotropic rod with insulated sides has initial temperature distribution U_Lx/L, $0 \le x \le L$ (U_L a constant). For time $t > 0$, its ends $x = 0$ and $x = L$ are held at temperatures 0°C and U_L°C, respectively. Find the temperature distribution in the rod for $t > 0$.

14. Repeat Exercise 13 if the initial temperature distribution is $f(x) = ax$, $0 \le x \le L$, where a is a constant. The ends $x = 0$ and $x = L$ are held at constant temperatures U_0°C and U_L°C, respectively, for $t > 0$.

15. Solve Exercise 5 in Section 4.3 in the case that $k \neq L^2/(n^2\pi^2)$ for any positive integer n. (See also Exercise 8 in Section 7.2.)

16. Solve Exercise 16 in Section 4.3 with zero initial temperature.

Part B Vibrations

17. A taut string has its ends fixed at $x = 0$ and $x = L$ on the x-axis. If it is given an initial displacement (at time $t = 0$) of $f(x) = kx(L - x)$, (k a constant), and zero initial velocity, find its subsequent displacement.

18. Solve Exercise 19(a) in Section 4.2.

19. Repeat Exercise 17 for zero initial displacement and an unspecified initial velocity $g(x)$.

20. Solve Exercise 37(a) in Section 7.2.

21. Solve Exercise 21 in Section 4.3.

22. Solve Exercise 21(a) in Section 4.2.

For Exercises 23–28 solve Exercises 30–35 in Section 7.2.

29. Repeat Example 10.14 if gravity is taken into account. See also Exercise 41 in Section 7.2.

30. Solve Exercise 28 in Section 7.2.

31. Show that Laplace transforms lead to the solution in part (c) for the problem in Exercise 20 of Section 4.3.

32. (a) Find a series solution for displacements in the bar of Exercise 24 of Section 7.2 if the constant force per unit area F is replaced by an impulse force $F = F_0\delta(t)$. Use the fact that

$$\int_0^\infty f(t)\delta(t)\,dt = f(0+).$$

(b) Show that the displacement of the end $x = L$ is $cF_0/(AE)$ times the square wave function

$$M_{2L/c}(t) = \begin{cases} 1, & 0 < t < 2L/c \\ -1, & 2L/c < t < 4L/c \end{cases}, \qquad M_{2L/c}(t + 4L/c) = M_{2L/c}(t).$$

33. Solve Exercise 42 in Section 7.2.

34. A taut string of length L is initially at rest along the x-axis. For time $t > 0$, its ends are subjected to prescribed displacements

$$y(0,t) = f_1(t), \qquad y(L,t) = f_2(t).$$

Find its displacement for $0 < x < L$ and $t > 0$.

35. (a) Show that the Laplace transform of the displacement function $y(x,t)$ for the vibrations in Exercise 45 of Section 7.2 is

$$\tilde{y}(x,s) = \frac{F_0\omega c \sinh{(sx/c)}}{s(s^2+\omega^2)[AE\cosh{(sL/c)} + mcs\sinh{(sL/c)}]}.$$

(b) Resonance occurs if either of the zeros $s = \pm i\omega$ of $s^2 + \omega^2$ coincides with a zero of

$$h(s) = AE\cosh(sL/c) + mcs\sinh(sL/c).$$

By expressing zeros of $h(s)$ in the form $s = c(\mu + \lambda i)$, show that

$$\tanh 2\mu L = \frac{-2AEmc^2\mu}{A^2E^2 + m^2c^4(\mu^2 + \lambda^2)}$$

and that therefore $\mu = 0$. Verify that resonance occurs if $\omega = c\lambda$ where λ is a root of the equation

$$\tan\lambda L = \frac{AE}{mc^2\lambda}.$$

36. Solve Example 4.4 in Section 4.2, but with an unspecified initial displacement $f(x)$. (Hint: Replace s by icq^2 in the ODE for $\tilde{y}(x, s)$.)

37. (a) The top of the bar in Exercise 20 is attached to a spring with constant k. If $x = 0$ corresponds to the top end of the bar when the spring is unstretched, show that the Laplace transform of the displacement function for cross sections of the bar is

$$\tilde{y}(x, s) = \frac{g}{s^3} - \frac{kgc\cosh[s(L-x)/c]}{s^3[AE\sinh(sL/c) + kc\cosh(sL/c)]}.$$

(b) Verify that $\tilde{y}(x, s)$ has a pole of order 1 at $s = 0$. What is the residue of $e^{st}\tilde{y}(x, s)$ at $s = 0$?

(c) By setting $s = c(\mu + \lambda i)$ to obtain zeros of

$$h(s) = AEs\sinh(sL/c) + kc\cosh(sL/c),$$

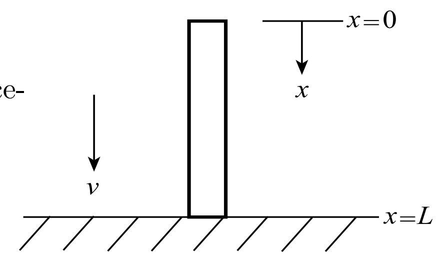
show that μ must be zero and that λ must satisfy

$$\tan\lambda L = \frac{k}{AE\lambda}.$$

(d) Find $y(x, t)$. (See also Exercise 38 in Section 7.2.)

38. (a) An unstrained elastic bar falls vertically under gravity with its axis in the vertical position (figure to the right). When its velocity is $v > 0$, it strikes a solid object and remains in contact with it thereafter. Show that the Laplace transform of displacements $y(x, t)$ of cross sections of the bar is

$$\tilde{y}(x, s) = \left(\frac{v}{s^2} + \frac{g}{s^3}\right)\left[1 - \frac{\cosh(sx/c)}{\cosh(sL/c)}\right].$$



(b) Use residues to find

$$y(x, t) = \frac{g(L^2 - x^2)}{2c^2} + \frac{8Lv}{\pi^2 c}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n-1)^2}\sin\frac{(2n-1)\pi ct}{2L}\cos\frac{(2n-1)\pi x}{2L}$$
$$+ \frac{16L^2 g}{\pi^3 c^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n-1)^3}\cos\frac{(2n-1)\pi ct}{2L}\cos\frac{(2n-1)\pi x}{2L}.$$

(c) Verify that the second series in part (b) may be expressed in the form

$$-\frac{g}{4c^2}[K(x+ct)+K(x-ct)],$$

where $K(x)$ is the even, odd-harmonic extension of $L^2 - x^2$, $0 \le x \le L$, to a function of period $4L$. (See Exercise 21 in Section 3.2 for the definition of an even, odd-harmonic function.)

(d) Verify that the first series in part (b) may be expressed in the form

$$\frac{v}{2c}[M_L(x+ct)-M_L(x-ct)],$$

where $M_L(x)$ is the odd, odd-harmonic extension of x, $0 \le x \le L$, to a function of period $4L$. (See Exercise 20 in Section 3.2 for the definition of an odd, odd-harmonic function.)

(e) Find an expression for the force $F(t)$ due to the bar on the cross section at $x = L$. Sketch graphs of $F(t)$ when $v < 2Lg/c$ and $v > 2gL/c$.

39. A bar 1/4 m long is falling as in Exercise 38 when it stikes an object squarely. Use the result of Exercise 38 to find a formula for the length of time of contact of the bar with the object. Use this formula to find the contact time for a steel bar with $\rho = 7.8 \times 10^3$ kg/m^3 and $E = 2.1 \times 10^{11}$ kg/m^2 when $v = 2$ m/s.