§6.6 Properties of Parabolic Partial Differential Equations

We now return to a difficulty posed in Chapter 4. In what sense are the series obtained in Chapters 4 and 6 "solutions" of their respective problems? In arriving at each series solution, we superposed an infinity of functions satisfying a linear, homogeneous PDE and linear, homogeneous boundary and/or initial conditions. Because of the questionable validity of this step (superposition principle 1 in Section 4.1 endorses only finite linear combinations), we have called each series a formal solution. It is now incumbent on us to verify that each formal solution is indeed a valid solution of its (initial) boundary value problem. Unfortunately, it is not possible to prove general results that encompass all problems solved by means of separation of variables and generalized Fourier series; on the other hand, the situation is not so bad that every problems is its own special case. Techniques exist that verify formal solutions for large classes of problems. In this section and Sections 6.7 and 6.8, we illustrate techniques that work when separation of variables leads to the Sturm-Liouville systems in Table 5.1. At the same time, we take the opportunity to develop properties of solutions of parabolic, hyperbolic and elliptic PDEs. Time-dependent heat conduction problems are manifested in parabolic equations; vibrations invariably involve hyperbolic equations; and potential problems give rise to elliptic equations.

We choose to illustrate the situation for parabolic PDEs with the heat conduction problem in equation 6.2 of Section 6.2,

$$\frac{\partial U}{\partial t} = k\frac{\partial^2 V}{\partial x^2}, \quad 0 < x < L, \quad t > 0, \tag{6.53a}$$

$$U_x(0,t) = 0, \quad t > 0, \tag{6.53b}$$

$$\kappa\frac{\partial U(L,t)}{\partial x} + \mu U(L,t) = 0, \quad t > 0, \tag{6.53c}$$

$$U(x,0) = f(x), \quad 0 < x < L. \tag{6.53d}$$

(See Exercise 1 for verification when both boundary conditions are Robin.) The formal solution of problem 6.53 is

$$U(x,t) = \sum_{n=1}^{\infty} c_n e^{-k\lambda_n^2 t} X_n(x) \qquad \text{where} \quad c_n = \int_0^L f(x)X_n(x)\,dx. \tag{6.54}$$

Eigenfunctions are $X_n(x) = N^{-1}\cos\lambda_n x$, where $2N^2 = L + (\mu/\kappa)/[\lambda_n^2 + (\mu/\kappa)^2]$, and eigenvalues are defined by the equation $\tan\lambda L = \mu/(\kappa\lambda)$.

We shall show by direct substitution that the function $U(x,t)$ defined by series 6.54 does indeed satisfy problem 6.53.

When coefficients c_n are calculated according to the formula in equation 6.54, the series $\sum_{n=1}^{\infty} c_n X_n(x)$ converges to $f(x)$ for $0 < x < L$ (provided $f(x)$ is piecewise smooth for $0 \le x \le L$). Since this series is $U(x,0)$, it follows that initial condition 6.53d is satisfied if $f(x)$ is piecewise smooth on $0 \le x \le L$, provided that at any point of discontinuity of $f(x)$, $f(x)$ is defined by $f(x) = [f(x+) + f(x-)]/2$.

To verify 6.53a–c, is not quite so simple. We first show that series 6.54 converges for all $0 \le x \le L$ and $t > 0$ and can be differentiated with respect to either x or t. Because eigenfunctions $X_n(x)$ are uniformly bounded (see Theorem 5.2 in Section 5.2), there exists a constant M such that for all $n \ge 1$ and $0 \le x \le L$,

$|X_n(x)| \leq N^{-1} \leq M$. Further, since $f(x)$ is piecewise continuous on $0 \leq x \leq L$, it is also bounded thereon; that is, $|f(x)| \leq K$, for some constant K. These two results imply that the coefficients c_n defined by 6.54 are bounded by

$$|c_n| \leq \int_0^L |f(x)||X_n(x)|\,dx \leq KML. \tag{6.55}$$

It follows that for any x in $0 \leq x \leq L$, and any time $t \geq t_0 > 0$,

$$\sum_{n=1}^{\infty} |c_n e^{-k\lambda_n^2 t} X_n(x)| \leq KM^2L \sum_{n=1}^{\infty} (e^{-kt_0})^{\lambda_n^2}.$$

Figure 5.3 indicates that the n^{th} eigenvalue $\lambda_n \geq (n-1)\pi/L$. Combine this with the fact that $e^{-kt_0} < 1$, and we may write, for $0 \leq x \leq L$ and $t \geq t_0 > 0$,

$$\begin{aligned}\sum_{n=1}^{\infty} |c_n e^{-k\lambda_n^2 t} X_n(x)| &\leq KM^2L \sum_{n=1}^{\infty} (e^{-kt_0})^{(n-1)^2\pi^2/L^2} \\ &\leq KM^2L \sum_{n=1}^{\infty} [(e^{-kt_0})^{\pi^2/L^2}]^{n-1} = KM^2L \sum_{n=1}^{\infty} r^{n-1},\end{aligned} \tag{6.56}$$

and the geometric series on the right converges, since $r = e^{-kt_0\pi^2/L^2} < 1$. According to the Weierstrass M-test (Theorem 3.3 in Section 3.3), series 6.54 converges absolutely and uniformly with respect to x and t for $0 \leq x \leq L$ and $t \geq t_0 > 0$. Because $t_0 > 0$ is arbitrary, it also follows that series 6.54 converges absolutely for $0 \leq x \leq L$ and $t > 0$.

Term-by-term differentiation of series 6.54 with respect to t gives

$$\sum_{n=1}^{\infty} -k\lambda_n^2 c_n e^{-k\lambda_n^2 t} X_n(x). \tag{6.57}$$

Since $\lambda_n \leq n\pi/L$ (see, once again, Figure 5.3), it follows that for all $0 \leq x \leq L$ and $t \geq t_0 > 0$,

$$\sum_{n=1}^{\infty} |-k\lambda_n^2 c_n e^{-k\lambda_n^2 t} X_n(x)| \leq \frac{kKM^2\pi^2}{L} \sum_{n=1}^{\infty} n^2 r^{n-1}. \tag{6.58}$$

Because the series $\sum_{n=1}^{\infty} n^2 r^{n-1}$ converges, we conclude that series 6.57 converges absolutely and uniformly with respect to x and t for $0 \leq x \leq L$ and $t \geq t_0 > 0$. As a result, series 6.57 represents $\partial U/\partial t$ for $0 \leq x \leq L$ and $t \geq t_0 > 0$. (Theorem 3.7 in Section 3.3). But, once again, the fact that t_0 is arbitrary implies that we may write

$$\frac{\partial U}{\partial t} = \sum_{n=1}^{\infty} -k\lambda_n^2 c_n e^{-k\lambda_n^2 t} X_n(x) \tag{6.59}$$

for $0 \leq x \leq L$ and $t > 0$.

Term-by-term differentiation of series 6.54 with respect to x gives

$$\sum_{n=1}^{\infty} c_n e^{-k\lambda_n^2 t} X_n'(x) = \sum_{n=1}^{\infty} c_n(-\lambda_n) e^{-k\lambda_n^2 t} N^{-1} \sin \lambda_n x. \tag{6.60}$$

Since $N^{-1} \le M$, we have, for $0 \le x \le L$ and $t \ge t_0 > 0$,

$$\sum_{n=1}^{\infty} |c_n e^{-k\lambda_n^2 t} X_n'(x)| \le \sum_{n=1}^{\infty} (KML)(\lambda_n M) e^{-k\lambda_n^2 t_0}$$
$$\le KM^2 L \sum_{n=1}^{\infty} \left(\frac{n\pi}{L}\right) r^{n-1} = KM^2 \pi \sum_{n=1}^{\infty} n r^{n-1}. \quad (6.61)$$

Because the series $\sum_{n=1}^{\infty} n r^{n-1}$ converges, series 6.60 likewise converges absolutely and uniformly. Consequently, series 6.54 may be differentiated term-by-term to yield, for $0 \le x \le L$ and $t > 0$,

$$\frac{\partial U}{\partial x} = \sum_{n=1}^{\infty} c_n e^{-k\lambda_n^2 t} X_n'(x). \quad (6.62)$$

A similar analysis shows that for $0 \le x \le L$ and $t > 0$,

$$\frac{\partial^2 U}{\partial x^2} = \sum_{n=1}^{\infty} c_n e^{-k\lambda_n^2 t} X_n''(x) = \sum_{n=1}^{\infty} c_n e^{-k\lambda_n^2 t} [-{\lambda_n}^2 X_n(x)]. \quad (6.63)$$

Expressions 6.59 and 6.63 for $\partial U/\partial t$ and $\partial^2 U/\partial x^2$ clearly indicate that $U(x, t)$ satisfies PDE 6.53a. Finally, expressions 6.62 and 6.54 for $\partial U/\partial x$ and $U(x, t)$ indicate that

$$\frac{\partial U(0,t)}{\partial x} = \sum_{n=1}^{\infty} c_n e^{-k\lambda_n^2 t} X_n'(0) = 0,$$

(since $X_n'(0) = 0$), and

$$\kappa \frac{\partial U(L,t)}{\partial x} + \mu U(L,t) = \kappa \sum_{n=1}^{\infty} c_n e^{-k\lambda_n^2 t} X_n'(L) + \mu \sum_{n=1}^{\infty} c_n e^{-k\lambda_n^2 t} X_n(L)$$
$$= \sum_{n=1}^{\infty} c_n e^{-k\lambda_n^2 t} [\kappa X_n'(L) + \mu X_n(L)] = 0,$$

(since $X_n(x)$ satisfies $\kappa X_n'(L) + \mu X_n(L) = 0$).

We have now verified that the formal solution $U(x, t)$ defined by series 6.54 satisfies equations 6.53a–d. Clearly demonstrated was the dependence of our verification on properties of the Sturm-Liouville system associated with 6.53. Indeed, indispensable were the facts that eigenvalues satisfied the inequalities $(n-1)\pi/L \le \lambda_n \le n\pi/L$ and that eigenfunctions were uniformly bounded. Without a knowledge of these properties, verification of the formal solution would have been impossible. Although series 6.54 satisfies problem 6.53, verification of 6.54 as the solution of the heat conduction problem described by 6.53 is not complete. To illustrate why, consider the function defined by

$$U(x,t) = \begin{cases} \displaystyle\sum_{n=1}^{\infty} b_n e^{-k\lambda_n^2 t} X_n(x), & 0 \le x \le L, \quad t > 0 \\ f(x), & 0 \le x \le L, \quad t = 0 \end{cases}, \quad (6.64)$$

where $\{b_n\}$ is a completely arbitrary, but bounded, sequence and $X_n(x)$ are the eigenfunctions in 6.54. The above procedure can once again be used to verify that function 6.64 also satisfies 6.53a–c; in addition, it satisfies 6.53d. This means that, as stated, problem 6.53 is not well posed; it does not have a unique solution. It cannot therefore be an adequate description of the physical problem following equation 6.2 in Section 6.2 — temperature in a rod of uniform cross section and insulated sides that at time $t = 0$ has temperature $f(x)$. For time $t > 0$, the end $x = 0$ is also insulated and heat is exchanged at the other end with an environment at temperature zero. In actual fact, 6.53 does have a unique solution, provided we demand that the solution satisfy certain continuity conditions. Our immediate objective, then, is to discover what these conditions are; once we find them, we can then verify that 6.54 is the one and only solution of 6.53.

Continuity conditions for $U(x,t)$ depend on the class of functions permitted for $f(x)$. To simplify discussions, suppose we permit only functions $f(x)$ that are continuous for $0 \le x \le L$ and have piecewise continuous first derivatives. Physically this is realistic; continuity of $f(x)$ implies that the initial temperature distribution in the rod must be continuous. Because $f'(x)$ is proportional to heat flux across cross sections of the rod, piecewise continuity of $f'(x)$ implies that initially there can be no infinite surges of heat.

With $f(x)$ continuous, it is reasonable, physically, to demaind that $U(x,t)$ be continuous for $0 \le x \le L$ and $t \ge 0$. (Were $f(x)$ assumed only piecewise continuous, continuity of $U(x,t)$ for $t = 0$ would be inappropriate.) The fact that $U(x,t)$ must satisfy PDE 6.53a suggests that we demand that $\partial U/\partial t$, $\partial U/\partial x$, and $\partial^2 U/\partial x^2$ all be continuous for $0 < x < L$ and $t > 0$. Boundary conditions 6.53b,c suggest that we require continuity of $\partial U/\partial x$ for $x = 0$, $t > 0$ and for $x = L$, $t > 0$ also. Because there are no heat sources (or sinks) at the ends of the rod, it follows that $\partial U/\partial t$ should be continuous at $x = 0$ and $x = L$ for $t > 0$. For a similar reason, $\partial^2 U/\partial x^2$ should also be continuous at $x = 0$ and $x = L$ for $t > 0$. We now show that these conditions guarantee a unique solution of problem 6.53; that is, we show that (when $f(x)$ is continuous and $f'(x)$ is piecewise continuous for $0 \le x \le L$) there is one and only one solution $U(x,t)$ of 6.53 that also satisfies

$$U(x,t) \text{ be continuous for } 0 \le x \le L \text{ and } t \ge 0; \tag{6.53e}$$

$$\frac{\partial U}{\partial x},\quad \frac{\partial U}{\partial t},\quad \frac{\partial^2 U}{\partial x^2} \text{ be continuous for } 0 \le x \le L \text{ and } t > 0. \tag{6.53f}$$

Suppose to the contrary, that there exist two solutions $U_1(x,t)$ and $U_2(x,t)$ satisfying 6.53a–f. The difference $U(x,t) = U_1(x,t) - U_2(x,t)$ must also satisfy 6.53a,b,c,e,f, but initial condition 6.53d is replaced by the homogeneous condition $U(x,0) = 0$, $0 < x < L$. To show that $U_1(x,t) \equiv U_2(x,t)$, we show that $U(x,t) \equiv 0$. To do this, we multiply PDE 6.53a by $U(x,t)$ and integrate with respect to x from $x = 0$ to $x = L$,

$$\int_0^L \frac{\partial U}{\partial t} U(x,t)\,dx = k\int_0^L \frac{\partial^2 U}{\partial x^2} U(x,t)\,dx, \quad t > 0.$$

Integration by parts on the right gives, for $t > 0$,

$$0 = \int_0^L \frac{1}{2}\frac{\partial [U(x,t)]^2}{\partial t} dx - k\left\{U(x,t)\frac{\partial U}{\partial x}\right\}_0^L + k\int_0^L \left(\frac{\partial U}{\partial x}\right)^2 dx$$

$$= \frac{1}{2}\int_0^L \frac{\partial(U^2)}{\partial t}dx - kU(L,t)\frac{\partial U(L,t)}{\partial x} + kU(0,t)\frac{\partial U(0,t)}{\partial x} + k\int_0^L \left(\frac{\partial U}{\partial x}\right)^2 dx. \quad (6.65)$$

Substitutions from boundary conditions 6.53b,c yield

$$0 = \frac{1}{2}\int_0^L \frac{\partial(U^2)}{\partial t}dx + k\int_0^L \left(\frac{\partial U}{\partial x}\right)^2 dx + \frac{k\mu[U(L,t)]^2}{\kappa}, \quad t > 0. \quad (6.66)$$

Because the last two terms are clearly nonnegative, we must have

$$\int_0^L \frac{\partial(U^2)}{\partial t}dx = \frac{\partial}{\partial t}\int_0^L [U(x,t)]^2 dx \le 0, \quad t > 0;$$

that is, the definite integral of $[U(x,t)]^2$ must be a nonincreasing function of t. But, because $U(x,t)$ satisfies the condition $U(x,0) = 0$, $0 < x < L$, the definite integral of $[U(x,t)]^2$ at $t = 0$ has value zero,

$$\int_0^L [U(x,0)]^2 dx = 0.$$

In other words, as a function of t, for $t \ge 0$, the definite integral of $[U(x,t)]^2$ is nonnegative, is nonincreasing, and has value zero at $t = 0$. It must therefore be identically equal to zero:

$$\int_0^L [U(x,t)]^2 dx \equiv 0, \quad t \ge 0.$$

Because the integrand is continuous and nonnegative, we conclude that $U(x,t) \equiv 0$ for $0 \le x \le L$ and $t \ge 0$; that is, $U_1(x,t) \equiv U_2(x,t)$.

We have shown then, that for the class of initial temperature distributions $f(x)$ that are continuous and have piecewise continuous first derivatives, conditions 6.53e,f attached to 6.53a–d yield a problem with a unique solution; there is one and only one solution satisfying 6.53a–f. To establish that 6.54 is the one and only one solution of problem 6.53, we must verify that it satisfies 6.53e,f. In verifying 6.54 as a solution of 6.53a–d, we proved that series 6.59, 6.62, and 6.63 converge uniformly for $0 \le x \le L$ and $t \ge t_0 > 0$ for arbitrary t_0. This implies that $\partial U/\partial t$, $\partial U/\partial x$, and $\partial^2 U/\partial x^2$ are all continuous functions for $0 \le x \le L$ and $t > 0$ (see Theorem 3.5 in Section 3.3). This establishes 6.53f. To verify 6.53e, we assume, for simplicity, that $f(x)$ satisfies the boundary conditions of the Sturm-Liouville system associated with the problem, namely $f'(0) = 0$ and $\kappa f'(L) + \mu f(L) = 0$. In this case, Theorem 5.4 in Section 5.3 indicates that the generalized Fourier series $\sum_{n=1}^{\infty} c_n X_n(x)$ of $f(x)$ converges uniformly to $f(x)$ for $0 \le x \le L$. Because the functions $e^{-k\lambda_n^2 t}$ are uniformly bounded for $t \ge 0$ and for each such t, the sequence $\{e^{-k\lambda_n^2 t}\}$ is nonincreasing, it follows by Abel's test (Theorem 3.4 in Section 3.3) that series 6.54 converges uniformly for $0 \le x \le L$ and $t \ge 0$. The temperature function $U(x,t)$ as defined by 6.54 must therefore be continuous for $0 \le x \le L$ and $t \ge 0$.

Verification of 6.54 as the solution to the heat conduction problem described by 6.53 is now complete.

An important point to notice here is that even though the initial temperature distribution may have discontinuities in its first derivative $f'(x)$, the solution of problem 6.53 has continuous first derivatives for $0 \le x \le L$ and $t > 0$. In fact, it has continuous derivative of all orders for $0 \le x \le L$ and $t > 0$. This means that the heat equation immediately smooths out discontinuities of $f'(x)$ and its derivatives. Even if $f(x)$ itself were piecewise continuous, discontinuities would immediately be smoothed out by the heat equation. We shall see that this is also true for elliptic equations, but not for hyperbolic ones.

The method used to verify that problem 6.53a–f has a unique solution is applicable to much more general problems. Consider, for example, the three-dimensional heat conduction problem

$$\frac{\partial U}{\partial t} = k\nabla^2 U + \frac{kg(x,y,z,t)}{\kappa}, \quad (x,y,z) \text{ in } V, \quad t > 0, \tag{6.67a}$$

$$U(x,y,z,t) = F(x,y,z,t), \quad (x,y,z) \text{ on } \beta(V), \quad t > 0, \tag{6.67b}$$

$$U(x,y,z,0) = f(x,y,z), \quad (x,y,z) \text{ in } V. \tag{6.67c}$$

In Exercise 2 it is proved that there cannot be more than one solution $U(x,y,z,t)$ that satisfies the conditions

$U(x,y,z,t)$ be continuous for (x,y,z) in $\overline{V}$ and $t \ge 0$, (6.67d)

First partial derivatives of $U(x,y,z,t)$ with respect to x, y, z, and t and second partial derivatives with respect to x, y, and z be continuous for (x,y,z) in $\overline{V}$ and $t > 0$, (6.67e)

where $\overline{V}$ is the closed region consisting of V and its boundary $\beta(V)$.

Heat conduction problems satisfy what are called maximum and minimum principles. We state and prove the one-dimensional situation here; three-dimensional principles are proved in Exercise 5. Temperature in a rod with insulated sides, when there is no internal heat generation and when the initial temperature distribution is $f(x)$, must satisfy the one-dimensional heat equation

$$\frac{\partial U}{\partial t} = k\frac{\partial^2 U}{\partial x^2}, \quad 0 < x < L, \quad t > 0, \tag{6.68a}$$

and the initial condition

$$U(x,0) = f(x), \quad 0 \le x \le L. \tag{6.68b}$$

By taking a closed interval in 6.68b, we are assuming compatibility between the initial temperature distribution $f(x)$ at $x = 0$ and $x = L$ and the boundary temperatures when $t = 0$. Boundary conditions have not been enunciated because maximum and minimum principles are independent of boundary conditions being Dirichlet, Neumann, or Robin. Let U_M be the largest of the following three numbers:

$$U_1 = \text{maximum value of } f(x) \text{ for } 0 \le x \le L,$$
$$U_2 = \text{maximum value of } U(0,t) \text{ for } 0 \le t \le T,$$
$$U_3 = \text{maximum value of } U(L,t) \text{ for } 0 \le t \le T,$$

where T is some given value of t. In other words, U_M is the maximum of the initial temperature of the rod and that found (or applied) at the ends of the rod up to time T. The **maximum principle** states that $U(x,t) \leq U_M$ for all $0 \leq x \leq L$ and $0 \leq t \leq T$; that is, at no point in the rod during the time interval $0 \leq t \leq T$ can the temperature ever exceed U_M. To prove this result, we define a function $V(x,t) = U(x,t) + \epsilon x^2$, $0 \leq x \leq L$, $0 \leq t \leq T$, where $\epsilon > 0$ is a very small number. Because U satisfies PDE 6.68a, we can say that for $0 < x < L$ and $0 < t < T$,

$$\frac{\partial V}{\partial t} - k\frac{\partial^2 V}{\partial x^2} = \frac{\partial U}{\partial t} - k\left(\frac{\partial^2 U}{\partial x^2} + 2\epsilon\right) = -2k\epsilon < 0. \tag{6.69}$$

Assuming that $U(x,t)$ is continuous, so also is $V(x,t)$, and therefore $V(x,t)$ must take on a maximum in the closed rectangle $\overline{A}$ of Figure 6.10. This value must occur either on the edge of the rectangle or at an interior point (x^*, t^*). In the latter case, $V(x,t)$ must necessarily have a relative maximum at (x^*, t^*), and therefore $\partial V/\partial t = \partial V/\partial x = 0$ and $\partial^2 V/\partial x^2 \leq 0$ at (x^*, t^*). But then $\partial V/\partial t - k\partial^2 V/\partial x^2 \geq 0$ at (x^*, t^*), contradicting inequality 6.69. Hence, the maximum value of V must occur on the boundary of $\overline{A}$. It cannot occur along $t = T$, for, in this case, $\partial V/\partial t \geq 0$ at the point and $\partial^2 V/\partial x^2$ would still be nonpositive. Once again, inequality 6.69 would be violated. Consequently, the maximum value of V on $\overline{A}$ must occur on one of the three boundaries $t = 0$, $x = 0$, or $x = L$. Since $U \leq U_M$ on these three lines, it follows that $V \leq U_M + \epsilon L^2$ on these lines and therefore in $\overline{A}$. But because $U(x,t) \leq V(x,t)$, we can state that, in $\overline{A}$, $U(x,t) \leq U_M + \epsilon L^2$. Since ϵ can be made arbitrarily small, it follows that U_M must be the maximum value of U for $0 \leq x \leq L$ and $0 \leq t \leq T$.

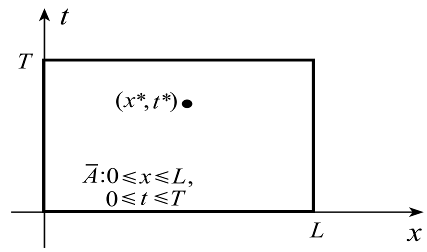

Figure 6.10

When this result is applied to $-U$, the **minimum principle** is obtained — at no point in the rod during the interval $0 \leq t \leq T$ can the temperature ever be less than the minimum of the initial temperature of the rod and that found (or applied) at the ends of the rod up to time T.

We mention one final property of heat conduction problems, which, unfortunately, is not demonstrable with the series solutions of Chapters 4 and 6. (It is illustrated for infinite rods in Case 2 of solution 11.35 in Section 11.4 and for finite rods in solution 10.44 of Section 10.4.) When heat is added to any part of an object, its effect is instantaneously felt throughout the whole object. For instance, suppose that the initial temperature $f(x)$ of the rod in problem 6.53 is identically equal to zero, and at $t = 0$ a small amount of heat is added to either end of the rod or over some cross section of the rod. Instantaneously, the temperature of every point of the rod rises. The increase may be extremely small, but nonetheless, every point in the rod has a positive temperature for arbitrarily small $t > 0$, and this is true for arbitrarily large L. In other words, heat has been propagated infinitely fast from the source point to all other points in the rod. This is a result of the macroscopic derivation of the heat equation in Section 2.2. On a microscopic level, it would be necessary to take into account the moment of inertia of the molecules transmitting heat, and this would lead to a finite speed for propagation of heat.

EXERCISES 6.6

1. (a) What is the formal series solution of the one-dimensional heat conduction problem

$$\frac{\partial U}{\partial t} = k\frac{\partial^2 V}{\partial x^2}, \quad 0 < x < L, \quad t > 0,$$
$$-l_1\frac{\partial U}{\partial x} + h_1 U = 0, \quad x = 0, \quad t > 0,$$
$$l_2\frac{\partial U}{\partial x} + h_2 U = 0, \quad x = L, \quad t > 0,$$
$$U(x,0) = f(x), \quad 0 < x < L?$$

(b) Use a technique similar to verification of formal solution 6.54 for problem 6.53 to verify that the formal solution in part (a) satisfies the four equations there when $f(x)$ is piecewise smooth on $0 \le x \le L$.

(c) Assuming further that $f(x)$ is continuous on $0 \le x \le L$, show that there is one and only one solution of the problem in part (a) that also satisfies continuity conditions 6.53e,f.

(d) Verify that the formal solution in part (a) satisfies 6.53e,f when $f(x)$ satisfies the boundary conditions of the associated Sturm-Liouville system.

2. Use Green's first identity (see Appendix C) to verify that there cannot be more than one solution to problem 6.67.

3. Repeat Exercise 2 if the boundary condition on $\beta(V)$ is of Robin type.

4. Can you repeat Exercise 2 if the boundary condition on $\beta(V)$ is of Neumann type?

5. In this exercise we prove three-dimensional maximum and minimum principles. Let $U(x,y,z,t)$ be the continuous solution of the homogeneous three-dimensional heat conduction equation in some open region V,

$$\frac{\partial U}{\partial t} = k\nabla^2 U, \quad (x,y,z) \text{ in } V, \quad t > 0,$$

which also satisfies the initial condition

$$U(x,y,z,0) = f(x,y,z), \quad (x,y,z) \text{ in } \overline{V},$$

where $\overline{V}$ is the closed region consisting of V and its boundary $\beta(V)$. Let U_M be the maximum value of $f(x,y,z)$ and the value of U on $\beta(V)$ for $0 \le t \le T$, T some given time.

(a) Define a function $W(x,y,z,t) = U(x,y,z,t) + \epsilon(x^2+y^2+z^2)$, where $\epsilon > 0$ is a very small number. Show that

$$\frac{\partial W}{\partial t} - k\nabla^2 W < 0$$

for (x,y,z) in V and $0 < t < T$, and use this fact to verify that W cannot have a relative maximum for a point (x,y,z) in V and a time $0 < t < T$.

(b) Prove the maximum principle that $U(x,y,z,t) \le U_M$ for (x,y,z) in $\overline{V}$ and $0 \le t \le T$.

(c) What is the minimum principle for this situation?