CHAPTER 9 PROBLEMS IN POLAR, CYLINDRICAL AND SPHERICAL COORDINATES

§9.1 Homogeneous Problems in Polar, Cylindrical, and Spherical Coordinates

In Section 6.3, separation of variables was used to solve homogeneous boundary value problems expressed in polar coordinates. With the results of Chapter 8, we are in a position to tackle boundary value problems in cylindrical and spherical coordinates and initial boundary value problems in all three coordinate systems. Homogeneous problems are discussed in this section; nonhomogeneous problems are discussed in Section 9.2.

We begin with the following heat conduction problem.

Example 9.1 An infinitely long cylinder of radius a is initially at temperature $f(r) = a^2 - r^2$, and for time $t > 0$, the boundary $r = a$ is insulated. Find the temperature in the cylinder for $t > 0$.

Solution With the initial temperature a function of r and the surface of the cylinder insulated, temperature in the cylinder is a function $U(r,t)$ of r and t only. It satisfies the initial boundary value problem

$$\frac{\partial U}{\partial t} = k\left(\frac{\partial^2 U}{\partial r^2} + \frac{1}{r}\frac{\partial U}{\partial r}\right), \quad 0 < r < a, \quad t > 0, \tag{9.1a}$$

$$\frac{\partial U(a,t)}{\partial r} = 0, \quad t > 0, \tag{9.1b}$$

$$U(r,0) = a^2 - r^2, \quad 0 < r < a. \tag{9.1c}$$

When a function $U(r,t) = R(r)T(t)$ with variables separated is substituted into the PDE, and the equation is divided by kRT, the result is

$$\frac{T'}{kT} = \frac{R''}{R} + \frac{R'}{rR} = \alpha = \text{constant independent of } r \text{ and } t.$$

This equation and boundary condition 9.1b yield the Sturm-Liouville system

$$(rR')' - \alpha r R = 0, \quad 0 < r < a, \tag{9.2a}$$

$$R'(a) = 0. \tag{9.2b}$$

This singular system was discussed in Section 8.4 (see Table 8.1 with $\nu = 0$). If we set $\alpha = -\lambda^2$, eigenvalues are defined by the equation $J_1(\lambda a) = 0$, and normalized eigenfunctions are

$$R_n(r) = \frac{\sqrt{2}J_0(\lambda_n r)}{aJ_0(\lambda_n a)}, \quad n \geq 0. \tag{9.3}$$

(For simplicity of notation, we have dropped the zero subscript on R_{0n} and λ_{0n}.)

The differential equation

$$T' + k\lambda_n^2 T = 0 \tag{9.4}$$

has general solution

$$T(t) = Ce^{-k\lambda_n^2 t}. \tag{9.5}$$

In order to satisfy initial condition 9.1c, we superpose separated functions and take

$$U(r,t) = \sum_{n=0}^{\infty} C_n e^{-k\lambda_n^2 t} R_n(r), \tag{9.6}$$

where the C_n are constants. Condition 9.1c requires these constants to satisfy

$$a^2 - r^2 = \sum_{n=0}^{\infty} C_n R_n(r), \quad 0 < r < a. \tag{9.7}$$

Thus, the C_n are coefficients in the Fourier Bessel series of $a^2 - r^2$, and, according to equation 8.62 in Section 8.4,

$$C_n = \int_0^a r(a^2 - r^2) R_n(r)\, dr = \frac{\sqrt{2}}{aJ_0(\lambda_n a)} \int_0^a r(a^2 - r^2) J_0(\lambda_n r)\, dr.$$

To evaluate this integral when $n > 0$, we set $u = \lambda_n r$, in which case

$$\begin{aligned} C_n &= \frac{\sqrt{2}}{aJ_0(\lambda_n a)} \int_0^{\lambda_n a} \left(\frac{a^2 u}{\lambda_n} - \frac{u^3}{\lambda_n^3} \right) J_0(u) \frac{du}{\lambda_n} \\ &= \frac{\sqrt{2}}{\lambda_n^4 a J_0(\lambda_n a)} \int_0^{\lambda_n a} (a^2 \lambda_n^2 u - u^3) J_0(u)\, du. \end{aligned}$$

For the term involving u^3, we use the reduction formula in Exercise 9 of Section 8.3,

$$\begin{aligned} C_n = \frac{\sqrt{2}}{\lambda_n^4 a J_0(\lambda_n a)} \Bigg[& a^2 \lambda_n^2 \int_0^{\lambda_n a} u J_0(u)\, du - \Big\{ u^3 J_1(u) \Big\}_0^{\lambda_n a} \\ & - \Big\{ 2u^2 J_0(u) \Big\}_0^{\lambda_n a} + 4 \int_0^{\lambda_n a} u J_0(u)\, du \Bigg]. \end{aligned}$$

If we recall the eigenvalue equation $J_1(\lambda a) = 0$, and equation 8.40 in Section 8.3 with $\nu = 1$, we may write

$$\begin{aligned} C_n &= \frac{\sqrt{2}}{\lambda_n^4 a J_0(\lambda_n a)} \left[-2\lambda_n^2 a^2 J_0(\lambda_n a) + (a^2 \lambda_n^2 + 4) \int_0^{\lambda_n a} \frac{d}{du} [u J_1(u)]\, du \right] \\ &= \frac{\sqrt{2}}{\lambda_n^4 a J_0(\lambda_n a)} \left[-2\lambda_n^2 a^2 J_0(\lambda_n a) + (a^2 \lambda_n^2 + 4) \Big\{ u J_1(u) \Big\}_0^{\lambda_n a} \right] \\ &= \frac{-2\sqrt{2}a}{\lambda_n^2}. \end{aligned}$$

When $n = 0$, the eigenfunction is $R_0(r) = \sqrt{2}/a$, and

$$C_0 = \int_0^a r(a^2 - r^2) R_0(r)\, dr = \frac{\sqrt{2}}{a} \left\{ \frac{a^2 r^2}{2} - \frac{r^4}{4} \right\}_0^a = \frac{\sqrt{2}a^3}{4}.$$

The solution of problem 9.1 is therefore

$$U(r,t) = \frac{\sqrt{2}a^3}{4}\left(\frac{\sqrt{2}}{a}\right) + \sum_{n=1}^{\infty}\frac{-2\sqrt{2}a}{\lambda_n^2}e^{-k\lambda_n^2 t}\frac{\sqrt{2}J_0(\lambda_n r)}{aJ_0(\lambda_n a)}$$
$$= \frac{a^2}{2} - 4\sum_{n=1}^{\infty}\frac{e^{-k\lambda_n^2 t}}{\lambda_n^2}\frac{J_0(\lambda_n r)}{J_0(\lambda_n a)}. \tag{9.8}$$

Notice that for large t, the limit of this solution is $a^2/2$, and this is the average value of $a^2 - r^2$ over the circle $r \le a$.•

In the following heat conduction problem, we add angular dependence to the temperature function.

Example 9.2 An infinitely long rod with semicircular cross section is initially ($t = 0$) at a constant nonzero temperature throughout. For $t > 0$, its flat side is held at temperature 0°C while its round side is insulated. Find temperature in the rod for $t > 0$.

Solution Temperature in that half of the rod for which $x < 0$ in Figure 9.1 is identical to that in the half for which $x \ge 0$; no heat crosses the $x = 0$ plane. As a result, the temperature function $U(r,\theta,t)$ (and it is independent of z) must satisfy the initial boundary value problem

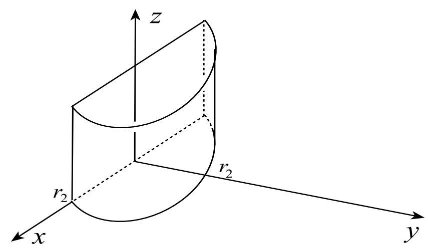


Figure 9.1

$$\frac{\partial U}{\partial t} = k\left(\frac{\partial^2 U}{\partial r^2} + \frac{1}{r}\frac{\partial U}{\partial r} + \frac{1}{r^2}\frac{\partial^2 U}{\partial \theta^2}\right), \quad 0 < r < a, \quad 0 < \theta < \frac{\pi}{2}, \quad t > 0, \tag{9.9a}$$
$$U(r,0,t) = 0, \quad 0 < r < a, \quad t > 0, \tag{9.9b}$$
$$U_\theta\left(r,\frac{\pi}{2},t\right) = 0, \quad 0 < r < a, \quad t > 0, \tag{9.9c}$$
$$U_r(a,\theta,t) = 0, \quad 0 < \theta < \frac{\pi}{2}, \quad t > 0, \tag{9.9d}$$
$$U(r,\theta,0) = U_0, \quad 0 < r < a, \quad 0 < \theta < \frac{\pi}{2}. \tag{9.9e}$$

(In Exercise 4, the problem is solved for $0 < \theta < \pi$ with the condition $U(r,\pi,t) = 0$ in place of 9.9c.)

When a function with variables separated, $U(r,\theta,t) = R(r)H(\theta)T(t)$, is substituted into the PDE,

$$RHT' = k(R''HT + r^{-1}R'HT + r^{-2}RH''T)$$

or,

$$-\frac{H''}{H} = \frac{r^2R''}{R} + \frac{rR'}{R} - \frac{r^2T'}{kT} = \alpha = \text{constant independent of } r,\ \theta, \text{ and } t.$$

When boundary conditions 9.9b,c are imposed on the separated function, a Sturm-Liouville system in $H(\theta)$ results,

$$H'' + \alpha H = 0, \quad 0 < \theta < \pi/2, \tag{9.10a}$$
$$H(0) = 0 = H'(\pi/2). \tag{9.10b}$$

This system was discussed in Section 5.2. If we set $\alpha = \nu^2$, then according to Table 5.1, eigenvalues are $\nu_m^2 = (2m-1)^2$ $(m = 1, 2, \ldots)$, with orthonormal eigenfunctions

$$H_m(\theta) = \frac{2}{\sqrt{\pi}} \sin(2m-1)\theta. \tag{9.11}$$

Continued separation of the equation in $R(r)$ and $T(t)$ gives

$$\frac{R'' + r^{-1}R'}{R} - \frac{\nu_m^2}{r^2} = \frac{T'}{kT} = \beta = \text{constant independent of } r \text{ and } t.$$

Boundary condition 9.9d leads to the Sturm-Liouville system

$$(rR')' + \left[-\beta r - \frac{(2m-1)^2}{r}\right] R = 0, \quad 0 < r < a, \tag{9.12a}$$

$$R'(a) = 0. \tag{9.12b}$$

This is singular Sturm-Liouville system 8.46 of Section 8.4. If we set $\beta = -\lambda^2$, eigenvalues λ_{mn} are defined by the equation

$$J'_{2m-1}(\lambda a) = 0 \tag{9.13}$$

with corresponding eigenfunctions

$$R_{mn}(r) = \frac{1}{N} J_{2m-1}(\lambda_{mn} r), \tag{9.14a}$$

where

$$2N^2 = a^2 \left[1 - \left(\frac{2m-1}{\lambda_{mn} a}\right)^2\right] [J_{2m-1}(\lambda_{mn} a)]^2. \tag{9.14b}$$

The differential equation

$$T' = -k\lambda_{mn}^2 T \tag{9.15}$$

has general solution

$$T(t) = Ce^{-k\lambda_{mn}^2 t}. \tag{9.16}$$

To satisfy initial condition 9.9e, we superpose separated functions and take

$$U(r, \theta, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} C_{mn} e^{-k\lambda_{mn}^2 t} R_{mn}(r) H_m(\theta), \tag{9.17}$$

where C_{mn} are constants. The initial condition requires these constants to satisfy

$$U_0 = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} C_{mn} R_{mn}(r) H_m(\theta), \quad 0 < r < a, \quad 0 < \theta < \pi/2. \tag{9.18}$$

If we multiply this equation by $H_i(\theta)$ and integrate with respect to θ from $\theta = 0$ to $\theta = \pi/2$, orthogonality of the eigenfunctions $H_m(\theta)$ gives

$$\sum_{n=1}^{\infty} C_{in} R_{in}(r) = \int_0^{\pi/2} U_0 H_i(\theta)\, d\theta = U_0 \int_0^{\pi/2} \frac{2}{\sqrt{\pi}} \sin(2i-1)\theta\, d\theta$$
$$= \frac{2U_0}{\sqrt{\pi}} \left\{ \frac{-1}{2i-1} \cos(2i-1)\theta \right\}_0^{\pi/2} = \frac{2U_0}{(2i-1)\sqrt{\pi}}.$$

But this equation implies that the C_{in} are Fourier Bessel coefficients for the function $2U_0/[(2i-1)\sqrt{\pi}]$; that is,

$$C_{in} = \int_0^a \frac{2U_0}{(2i-1)\sqrt{\pi}} r R_{in}(r)\, dr.$$

Thus, the solution of problem 9.9 for $0 \le \theta \le \pi/2$ is 9.17, where

$$C_{mn} = \frac{2U_0}{(2m-1)\sqrt{\pi}} \int_0^a r R_{mn}(r)\, dr. \tag{9.19}$$

For an angle θ between $\pi/2$ and π, we should evaluate $U(r, \pi - \theta, t)$. Since

$$H_m(\pi - \theta) = \frac{2}{\sqrt{\pi}} \sin(2m-1)(\pi - \theta) = \frac{2}{\sqrt{\pi}} \sin(2m-1)\theta,$$

it follows that $U(r, \pi - \theta, t) = U(r, \theta, t)$. Hence, solution 9.17 is valid for $0 \le \theta \le \pi$.•

Our next example is a vibration problem.

Example 9.3 Solve the initial boundary value problem

$$\frac{\partial^2 z}{\partial t^2} = c^2 \left(\frac{\partial^2 z}{\partial r^2} + \frac{1}{r}\frac{\partial z}{\partial r} + \frac{1}{r^2}\frac{\partial^2 z}{\partial \theta^2} \right), \quad 0 < r < a, \quad -\pi < \theta \le \pi, \quad t > 0, \tag{9.20a}$$
$$z(a, \theta, t) = 0, \quad -\pi < \theta \le \pi, \quad t > 0, \tag{9.20b}$$
$$z(r, \theta, 0) = f(r, \theta), \quad 0 < r < a, \quad -\pi < \theta \le \pi, \tag{9.20c}$$
$$z_t(r, \theta, 0) = 0, \quad 0 < r < a, \quad -\pi < \theta \le \pi. \tag{9.20d}$$

Described is a membrane stretched over the circle $r \le a$ that has an initial displacement $f(r, \theta)$ and zero initial velocity. Boundary condition 9.20b states that the edge of the membrane is fixed on the xy-plane.

Solution When a separated function $z(r, \theta, t) = R(r)H(\theta)T(t)$, is substituted into the PDE,

$$RHT'' = c^2(R''HT + r^{-1}R'HT + r^{-2}RH''T)$$

or,

$$-\frac{H''}{H} = r^2 \left(\frac{R'' + r^{-1}R'}{R} - \frac{T''}{c^2 T} \right) = \alpha = \text{constant independent of } r,\ \theta, \text{ and } t.$$

Since the solution and its first derivative with respect to θ must be 2π-periodic in θ, it follows that $H(\theta)$ must satisfy the periodic Sturm-Liouville system

$$H'' + \alpha H = 0, \quad -\pi < \theta \le \pi, \tag{9.21a}$$
$$H(-\pi) = H(\pi), \tag{9.21b}$$
$$H'(-\pi) = H'(\pi). \tag{9.21c}$$

This system was discussed in Chapter 5 (Example 5.2 and equation 5.20). Eigenvalues are $\alpha = m^2$, m a nonnegative integer, with orthonormal eigenfunctions

$$\frac{1}{\sqrt{2\pi}}, \quad \frac{1}{\sqrt{\pi}}\sin m\theta, \quad \frac{1}{\sqrt{\pi}}\cos m\theta. \tag{9.22}$$

Continued separation of the equation in $R(r)$ and $T(t)$ gives

$$\frac{R'' + r^{-1}R'}{R} - \frac{m^2}{r^2} = \frac{T''}{c^2T} = \beta = \text{constant independent of } r \text{ and } t.$$

When boundary condition 9.20b is imposed on the separated function, a Sturm-Liouville system in $R(r)$ results,

$$(rR')' + \left(-\beta r - \frac{m^2}{r}\right)R = 0, \quad 0 < r < a, \tag{9.23a}$$

$$R(a) = 0. \tag{9.23b}$$

This is, once again, singular system 8.46 in Section 8.4. If we set $\beta = -\lambda^2$, eigenvalues λ_{mn} are defined by

$$J_m(\lambda a) = 0, \tag{9.24}$$

with corresponding orthonormal eigenfunctions

$$R_{mn}(r) = \frac{\sqrt{2}J_m(\lambda_{mn}r)}{aJ_{m+1}(\lambda_{mn}a)} \tag{9.25}$$

(see Table 8.1). The differential equation

$$T'' + c^2\lambda_{mn}^2 T = 0 \tag{9.26}$$

has general solution

$$T(t) = d\cos c\lambda_{mn}t + b\sin c\lambda_{mn}t, \tag{9.27}$$

where d and b are constants. Initial condition 9.20d implies that $b = 0$, and hence

$$T(t) = d\cos c\lambda_{mn}t. \tag{9.28}$$

In order to satisfy the final initial condition 9.20c, we superpose separated functions and take

$$\begin{aligned} z(r,\theta,t) &= \sum_{n=1}^{\infty} d_{0n}\frac{R_{0n}(r)}{\sqrt{2\pi}}\cos c\lambda_{0n}t \\ &\quad + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} R_{mn}(r)\left(d_{mn}\frac{\cos m\theta}{\sqrt{\pi}} + f_{mn}\frac{\sin m\theta}{\sqrt{\pi}}\right)\cos c\lambda_{mn}t, \end{aligned} \tag{9.29}$$

where d_{mn} and f_{mn} are constants. Condition 9.20c requires these constants to satisfy

$$f(r,\theta) = \sum_{n=1}^{\infty} d_{0n}\frac{R_{0n}(r)}{\sqrt{2\pi}} + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} R_{mn}(r)\left(d_{mn}\frac{\cos m\theta}{\sqrt{\pi}} + f_{mn}\frac{\sin m\theta}{\sqrt{\pi}}\right) \tag{9.30}$$

for $0 < r < a$, $-\pi < \theta \le \pi$. If we multiply this equation by $(1/\sqrt{\pi})\cos i\theta$ and integrate with respect to θ from $\theta = -\pi$ to $\theta = \pi$, orthogonality of the eigenfunctions in θ gives

$$\int_{-\pi}^{\pi} f(r,\theta)\frac{\cos i\theta}{\sqrt{\pi}}d\theta = \sum_{n=1}^{\infty} d_{in}R_{in}(r).$$

Multiplication of this equation by $rR_{ij}(r)$ and integration with respect to r from $r = 0$ to $r = a$ yields (because of orthogonality of the R_{ij} for fixed i)

$$\int_0^a\int_{-\pi}^{\pi} rf(r,\theta)R_{ij}\frac{\cos i\theta}{\sqrt{\pi}}d\theta\,dr = d_{ij};$$

that is

$$d_{mn} = \int_{-\pi}^{\pi}\int_0^a rR_{mn}\frac{\cos m\theta}{\sqrt{\pi}}f(r,\theta)\,dr\,d\theta. \tag{9.31a}$$

Similarly,

$$f_{mn} = \int_{-\pi}^{\pi}\int_0^a rR_{mn}\frac{\sin m\theta}{\sqrt{\pi}}f(r,\theta)\,dr\,d\theta, \tag{9.31b}$$

and

$$d_{0n} = \int_{-\pi}^{\pi}\int_0^a rR_{0n}\frac{f(r,\theta)}{\sqrt{2\pi}}\,dr\,d\theta. \tag{9.31c}$$

The solution of problem 9.20 is therefore 9.29, where d_{mn} and f_{mn} are defined by 9.31.●

Coefficients d_{mn} and f_{mn} in this example were calculated by first using orthogonality of the trigonometric eigenfunctions and then using orthogonality of the $R_{mn}(r)$. An alternative procedure is to determine the multi-dimensional eigenfunctions for problem 9.20. This approach is discussed in Exercise 27.

Our final example on separation is a potential problem.

Example 9.4 Find the potential interior to a sphere when the potential is $f(\phi,\theta)$ on the sphere.

Solution The boundary value problem for the potential $V(r,\phi,\theta)$ is

$$\frac{\partial^2 V}{\partial r^2} + \frac{2}{r}\frac{\partial V}{\partial r} + \frac{1}{r^2\sin\phi}\frac{\partial}{\partial\phi}\left(\sin\phi\frac{\partial V}{\partial\phi}\right) + \frac{1}{r^2\sin^2\phi}\frac{\partial^2 V}{\partial\theta^2} = 0,$$
$$0 < r < a, \quad 0 < \phi < \pi, \quad -\pi < \theta \le \pi, \tag{9.32a}$$
$$V(a,\phi,\theta) = f(\phi,\theta), \quad 0 \le \phi \le \pi, \quad -\pi < \theta \le \pi. \tag{9.32b}$$

When a function with variables separated, $V(r,\phi,\theta) = R(r)\Phi(\phi)H(\theta)$, is substituted into PDE 9.32a,

$$R''\Phi H + \frac{2}{r}R'\Phi H + \frac{1}{r^2\sin\phi}\frac{\partial}{\partial\phi}(\sin\phi\, R\Phi' H) + \frac{R\Phi H''}{r^2\sin^2\phi} = 0$$

or,

$$r^2\sin^2\phi\left[\frac{R''}{R} + \frac{2R'}{rR} + \frac{1}{r^2\sin\phi\,\Phi}\frac{d}{d\phi}(\sin\phi\,\Phi')\right] = -\frac{H''}{H}$$
$$= \alpha = \text{constant independent of } r,\ \phi, \text{ and } \theta.$$

Because $V(r,\phi,\theta)$ must be 2π-periodic in θ, as must its first derivative with respect to θ, it follows that $H(\theta)$ must satisfy the periodic Sturm-Liouville system

$$H'' + \alpha H = 0, \quad -\pi < \theta \le \pi, \tag{9.33a}$$

$$H(-\pi) = H(\pi), \tag{9.33b}$$

$$H'(-\pi) = H'(\pi). \tag{9.33c}$$

This is Sturm-Liouville system 9.21 with eigenvalues $\alpha = m^2$ and orthonormal eigenfunctions

$$\frac{1}{\sqrt{2\pi}}, \qquad \frac{1}{\sqrt{\pi}}\cos m\theta, \qquad \frac{1}{\sqrt{\pi}}\sin m\theta.$$

Continued separation of the equation in $R(r)$ and $\Phi(\phi)$ gives

$$\frac{r^2R''}{R} + \frac{2rR'}{R} = \frac{m^2}{\sin^2\phi} - \frac{1}{\Phi\sin\phi}\frac{d}{d\phi}(\sin\phi\,\Phi') = \beta = \text{constant independent of } r \text{ and } \phi.$$

Thus, $\Phi(\phi)$ must satisfy the singular Sturm-Liouville system

$$\frac{d}{d\phi}\left(\sin\phi\frac{d\Phi}{d\phi}\right) + \left(\beta\sin\phi - \frac{m^2}{\sin\phi}\right)\Phi = 0, \quad 0 < \phi < \pi. \tag{9.34}$$

According to the results of Section 8.6, eigenvalues are $\beta = n(n+1)$, where $n \ge m$ is an integer, with orthonormal eigenfunctions

$$\Phi_{mn}(\phi) = \sqrt{\frac{(2n+1)(n-m)!}{2(n+m)!}}P_{mn}(\cos\phi). \tag{9.35}$$

The remaining differential equation

$$r^2R'' + 2rR' - n(n+1)R = 0 \tag{9.36}$$

is a Cauchy-Euler equation that can be solved by setting $R(r) = r^s$, s an unknown constant. This results in the general solution

$$R(r) = \frac{C}{r^{n+1}} + Ar^n. \tag{9.37}$$

For $R(r)$ to remain bounded as r approaches zero, we must set $C = 0$. Superposition of separated functions now yields

$$\begin{aligned} V(r,\phi,\theta) &= \sum_{n=0}^{\infty}\frac{1}{\sqrt{2\pi}}A_{0n}r^n\Phi_{0n}(\phi) \\ &\quad + \sum_{m=1}^{\infty}\sum_{n=m}^{\infty} r^n\Phi_{mn}(\phi)\left(A_{mn}\frac{\cos m\theta}{\sqrt{\pi}} + B_{mn}\frac{\sin m\theta}{\sqrt{\pi}}\right), \end{aligned} \tag{9.38}$$

where A_{mn} and B_{mn} are constants. Boundary condition 9.32b requires these constants to satisfy

$$\begin{aligned} f(\phi,\theta) &= \sum_{n=0}^{\infty}\frac{1}{\sqrt{2\pi}}A_{0n}a^n\Phi_{0n}(\phi) \\ &\quad + \sum_{m=1}^{\infty}\sum_{n=m}^{\infty} a^n\Phi_{mn}(\phi)\left(A_{mn}\frac{\cos m\theta}{\sqrt{\pi}} + B_{mn}\frac{\sin m\theta}{\sqrt{\pi}}\right) \end{aligned} \tag{9.39}$$

for $0 \le \phi \le \pi$, $-\pi < \theta \le \pi$. Because of orthogonality of eigenfunctions in ϕ and θ, multiplication by $(1/\sqrt{2\pi}) \sin\phi\, \Phi_{0j}(\phi)$ and integration with respect to ϕ and θ give

$$A_{0j} = \frac{1}{a^j} \int_{-\pi}^{\pi} \int_0^{\pi} f(\phi, \theta) \frac{1}{\sqrt{2\pi}} \sin\phi\, \Phi_{0j}(\phi)\, d\phi\, d\theta. \tag{9.40a}$$

Similarly,

$$A_{mn} = \frac{1}{a^n} \int_{-\pi}^{\pi} \int_0^{\pi} f(\phi, \theta) \frac{\cos m\theta}{\sqrt{\pi}} \sin\phi\, \Phi_{mn}(\phi)\, d\phi\, d\theta, \tag{9.40b}$$

$$B_{mn} = \frac{1}{a^n} \int_{-\pi}^{\pi} \int_0^{\pi} f(\phi, \theta) \frac{\sin m\theta}{\sqrt{\pi}} \sin\phi\, \Phi_{mn}(\phi)\, d\phi\, d\theta. \tag{9.40c}$$

Notice that the potential at the centre of the sphere is

$$V(0, \phi, \theta) = \frac{1}{\sqrt{2\pi}} A_{00} \Phi_{00}(\phi) = \frac{1}{\sqrt{2\pi}} \left[\int_{-\pi}^{\pi} \int_0^{\pi} f(\phi, \theta) \frac{1}{\sqrt{2\pi}} \sin\phi\, \Phi_{00}(\phi)\, d\phi\, d\theta \right] \Phi_{00}(\phi).$$

Since $\Phi_{00}(\phi) = 1/\sqrt{2}$,

$$\begin{aligned} V(0, \phi, \theta) &= \frac{1}{4\pi} \int_{-\pi}^{\pi} \int_0^{\pi} f(\phi, \theta) \sin\phi\, d\phi\, d\theta \\ &= \frac{1}{4\pi a^2} \int_{-\pi}^{\pi} \int_0^{\pi} f(\phi, \theta) a^2 \sin\phi\, d\phi\, d\theta, \end{aligned}$$

and this is the average value of $f(\phi, \theta)$ over the surface of the sphere. We can develop an integral formula for the solution analogous to Poisson's integral formula for a circle, equation 6.34. We change variables of integration for the coefficients to α and β, substitute the coefficients into summation 9.38 and interchange orders of integration and summation

$$\begin{aligned} V(r, \phi, \theta) = \int_0^{\pi} \int_{-\pi}^{\pi} \Bigg[&\frac{1}{2\pi} \sum_{n=0}^{\infty} \left(\frac{r}{a}\right)^n f(\alpha, \beta) \sin\alpha\, \Phi_{0n}(\phi) \Phi_{0n}(\alpha) \\ &+ \frac{1}{\pi} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left(\frac{r}{a}\right)^n f(\alpha, \beta) \sin\alpha\, \Phi_{mn}(\phi) \Phi_{mn}(\alpha) (\cos m\theta \cos m\beta + \sin m\theta \sin m\beta) \Bigg] d\beta\, d\alpha \\ = \frac{1}{\pi} \int_0^{\pi} \int_{-\pi}^{\pi} f(\alpha, \beta) \sin\alpha \Bigg[&\frac{1}{2} \sum_{n=0}^{\infty} \left(\frac{r}{a}\right)^n \Phi_{0n}(\phi) \Phi_{0n}(\alpha) \\ &+ \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left(\frac{r}{a}\right)^n \Phi_{mn}(\phi) \Phi_{mn}(\alpha) \cos m(\theta - \beta) \Bigg] d\beta\, d\alpha. \end{aligned}$$

Let us define

$$S(r, \phi, \theta) = \frac{1}{2} \sum_{n=0}^{\infty} \left(\frac{r}{a}\right)^n \Phi_{0n}(\phi) \Phi_{0n}(\alpha) + \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left(\frac{r}{a}\right)^n \Phi_{mn}(\phi) \Phi_{mn}(\alpha) \cos m(\theta - \beta).$$

Consider the potential at a point inside the sphere and on the z-axis with spherical coordinates $(r, 0, \theta)$, where θ is arbitrary and $0 < r < a$. For such a point,

$$S(r, 0, \theta) = \frac{1}{2} \sum_{n=0}^{\infty} \left(\frac{r}{a}\right)^n \Phi_{0n}(0) \Phi_{0n}(\alpha) + \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left(\frac{r}{a}\right)^n \Phi_{mn}(0) \Phi_{mn}(\alpha) \cos m(\theta - \beta).$$

Since

$$\Phi_{0n}(0) = \sqrt{\frac{2n+1}{2}}P_n(1) = \sqrt{\frac{2n+1}{2}}, \qquad \Phi_{mn}(0) = \sqrt{\frac{(2n+1)(n-m)!}{2(n+m)!}}P_{mn}(1) = 0,$$

$$S(r,0,\theta) = \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{r}{a}\right)^n\left(\frac{2n+1}{2}\right)P_n(\cos\alpha) = \frac{1}{4}\sum_{n=0}^{\infty}(2n+1)\left(\frac{r}{a}\right)^n P_n(\cos\alpha).$$

To find a closed value for this summation, we differentiate the generating function 8.72 for Legendre polynomials

$$\frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty}P_n(x)t^n$$

with respect to t,

$$\frac{x-t}{(1-2xt+t^2)^{3/2}} = \sum_{n=0}^{\infty}nP_n(x)t^{n-1}.$$

If we multiply this by $2t$ and add it to the generating function, we obtain

$$\sum_{n=0}^{\infty}(2n+1)P_n(x)t^n = \frac{2t(x-t)}{(1-2xt+t^2)^{3/2}} + \frac{1}{\sqrt{1-2xt+t^2}} = \frac{1-t^2}{(1-2xt+t^2)^{3/2}}.$$

It follows that

$$S(r,0,\theta) = \frac{1}{4}\left[\frac{1-\dfrac{r^2}{a^2}}{\left(1-\dfrac{2r\cos\alpha}{a}+\dfrac{r^2}{a^2}\right)^{3/2}}\right] = \frac{a(a^2-r^2)}{4(a^2-2ar\cos\alpha+r^2)^{3/2}}.$$

Thus,

$$\begin{aligned} V(r,0,\theta) &= \frac{1}{\pi}\int_0^{\pi}\int_{-\pi}^{\pi}\frac{a(a^2-r^2)}{4(a^2-2ar\cos\alpha+r^2)^{3/2}}f(\alpha,\beta)\sin\alpha\,d\beta\,d\alpha \\ &= \frac{a(a^2-r^2)}{4\pi}\int_0^{\pi}\int_{-\pi}^{\pi}\frac{f(\alpha,\beta)\sin\alpha}{(a^2-2ar\cos\alpha+r^2)^{3/2}}d\beta\,d\alpha. \end{aligned}$$

This is the potential at a point $(r,0,\theta)$ on the z-axis. The distance between this point and a point (a,α,β) on the sphere is

$$\sqrt{(a\sin\alpha\cos\beta)^2+(a\sin\alpha\sin\beta)^2+(a\cos\alpha-r)^2} = \sqrt{r^2+a^2-2ar\cos\alpha}.$$

The denominator in the above integral is therefore the cube of the distance from points on the sphere to the point at which the potential is calculated. Since the axes could always be rotated so that the observation point is on the z-axis, it follows that to find the potential at any point with spherical coordinates (r,ϕ,θ) inside the sphere, we need only replace $\sqrt{r^2+a^2-2ar\cos\alpha}$ with the distance from (r,ϕ,θ) to (a,α,β), namely,

$$\begin{aligned} &\sqrt{(r\sin\phi\cos\theta-a\sin\alpha\cos\beta)^2+(r\sin\phi\sin\theta-a\sin\alpha\sin\beta)^2+(r\cos\phi-a\cos\alpha)^2} \\ &\quad = \sqrt{r^2+a^2-2ar[\sin\phi\sin\alpha\cos(\theta-\beta)+\cos\phi\cos\alpha]}. \end{aligned}$$

Thus,

$$V(r,\phi,\theta) = \frac{a(a^2-r^2)}{4\pi}\int_0^\pi\int_{-\pi}^\pi \frac{f(\alpha,\beta)\sin\alpha}{\{r^2+a^2-2ar[\sin\phi\sin\alpha\cos(\theta-\beta)+\cos\phi\cos\alpha]\}^{3/2}}d\beta\,d\alpha. \tag{9.41}$$

This is called **Poisson's integral formula for a sphere.**•

EXERCISES 9.1

Part A Heat Conduction

1. (a) The initial temperature of an infinitely long cylinder of radius a is $f(r)$. If, for time $t > 0$, the outer surface is held at 0°C, find the temperature in the cylinder.
(b) Simplify the solution in part (a) when $f(r)$ is a constant U_0.
(c) Find the solution when $f(r) = a^2 - r^2$.

2. An infinitely long cylinder of radius a is initially at temperature $f(r)$ and, for time $t > 0$, the boundary $r = a$ is insulated.
(a) Find the temperature $U(r,t)$ in the cylinder.
(b) What is the limit of $U(r,t)$ for large t?

3. A thin circular plate of radius a is insulated top and bottom. At time $t = 0$ its temperature is $f(r,\theta)$. If the temperature of its edge is held at 0°C for $t > 0$, find its interior temperature for $t > 0$.

4. Solve Example 9.2 using the boundary condition $U(r,\pi,t) = 0$ in place of $\partial U(r,\pi/2,t)/\partial\theta = 0$.

5. An infinitely long cylinder is bounded by the surfaces $r = a$, $\theta = 0$, and $\theta = \pi/2$. At time $t = 0$, its temperature is $f(r,\theta)$, and for $t > 0$, all surfaces are held at temperature zero. Find temperature in the cylinder.

6. Repeat Exercise 5 if the flat sides are insulated.

7. Repeat Exercise 5 if the curved side is insulated.

8. Repeat Exercise 5 if all sides are insulated. Show that the limit of the temperature as $t \to \infty$ is the average of $f(r,\theta)$ over the cylinder.

9. A flat plate in the form of a sector of a circle of radius 1 and angle α is insulated top and bottom. At time $t = 0$, the temperature of the plate increases linearly from 0°C at $r = 0$ to a constant value $\bar{U}$°C at $r = 1$ (and is therefore independent of θ). If, for $t > 0$, the rounded edge is insulated and the straight edges are held at temperature 0°C, find the temperature in the plate for $t > 0$. Prove that heat never crosses the line $\theta = \alpha/2$.

10. Find the temperature in the plate of Exercise 9 if the initial temperature is $f(r)$, the straight sides are insulated, and the curved edge is held at temperature 0°C.

11. Repeat Exercise 10 if the initial temperature is a function of r and θ, namely, $f(r,\theta)$.

12. A cylinder occupies the region $r \le a$, $0 \le z \le L$. It has temperature $f(r,z)$ at time $t = 0$. For $t > 0$, its end $z = 0$ is insulated, and the remaining two surfaces are held at temperature 0°C. Find the temperature in the cylinder.

13. Solve Exercise 1(a),(b) if heat is transferred at $r = a$ according to Newton's law of cooling to an environment at temperature zero.

14. (a) A sphere of radius a is initially at temperature $f(r)$ and, for time $t > 0$, the boundary $r = a$ is held at temperature zero. Find the temperature in the sphere for $t > 0$. (You will need the results of Exercise 8 in Section 8.4). Compare the solution to that in Exercise 12 of Section 4.2.
(b) Simplify the solution when $f(r) = U_0$, a constant.
(c) Suppose the sphere has radius 20 cm and is made of steel with $k = 12.4 \times 10^{-6}$. Find the temperature at the centre of the sphere after 10 minutes when $f(r) = U_0$ as in part (b).
(d) Repeat part (c) if the sphere is asbestos with $k = 0.247 \times 10^{-6}$.

15. Repeat parts (a) and (b) of Exercise 14 if the surface of the sphere is insulated. (See Exercise 8 in Section 8.4.) What is the temperature for large t?

16. Repeat parts (a) and (b) of Exercise 14 if the surface transfers heat to an environment at temperature zero according to Newton's law of cooling; that is, take as boundary condition

$$\kappa \frac{\partial U(a,t)}{\partial r} + \mu U(a,t) = 0, \quad t > 0.$$

(Assume that $\mu a < \kappa$ and see Exercise 8 in Section 8.4.)

17. Repeat Exercise 14(a) if the initial temperature is also a function of ϕ. (You will need the results of Exercise 9 in Section 8.4.)

18. (a) Repeat Exercise 14(a) if the initial temperature is also a function of ϕ and the surface of the sphere is insulated. (You will need the results of Exercise 9 in Section 8.4.)
(b) What is the limit of the solution for large t?

19. The result of this exercise is analogous to that in Exercise 9 of Section 6.4. Show that the solution of the homogeneous heat conduction problem

$$\begin{aligned}
\frac{\partial U}{\partial t} &= k\left(\frac{\partial^2 U}{\partial r^2} + \frac{1}{r}\frac{\partial U}{\partial r} + \frac{\partial^2 U}{\partial z^2}\right), \quad 0 < r < a, \quad 0 < z < L, \quad t > 0,\\
-l_1 \frac{\partial U}{\partial z} &+ h_1 U = 0, \quad z = 0, \quad 0 < r < a, \quad t > 0,\\
l_2 \frac{\partial U}{\partial z} &+ h_2 U = 0, \quad z = L, \quad 0 < r < a, \quad t > 0,\\
l_3 \frac{\partial U}{\partial r} &+ h_3 U = 0, \quad r = a, \quad 0 < z < L, \quad t > 0,\\
U(r,z,0) &= f(r)g(z), \quad 0 < r < a, \quad 0 < z < L,
\end{aligned}$$

where the initial temperature is the product of a function of r and a function of z, is the product of the solutions of the problems

$$\begin{aligned}
\frac{\partial U}{\partial t} &= k\left(\frac{\partial^2 U}{\partial r^2} + \frac{1}{r}\frac{\partial U}{\partial r}\right), \quad 0 < r < a, \quad t > 0,\\
l_3 \frac{\partial U(a,t)}{\partial r} &+ h_3 U(a,t) = 0, \quad t > 0,\\
U(r,0) &= f(r), \quad 0 < r < a;
\end{aligned}$$

and

$$\frac{\partial U}{\partial t} = k\frac{\partial^2 U}{\partial z^2}, \quad 0 < z < L, \quad t > 0,$$
$$-l_1\frac{\partial U(0,t)}{\partial z} + h_1 U(0,t) = 0, \quad t > 0,$$
$$l_2\frac{\partial U(L,t)}{\partial z} + h_2 U(L,t) = 0, \quad t > 0,$$
$$U(z,0) = g(z), \quad 0 < z < L.$$

20. Solve the heat conduction problem

$$\frac{\partial U}{\partial t} = k\left(\frac{\partial^2 U}{\partial r^2} + \frac{1}{r}\frac{\partial U}{\partial r} + \frac{\partial^2 U}{\partial z^2}\right), \quad 0 < r < a, \quad 0 < z < L, \quad t > 0,$$
$$U_z(r,0,t) = 0, \quad 0 < r < a, \quad t > 0,$$
$$U(r,L,t) = 0, \quad 0 < r < a, \quad t > 0,$$
$$U_r(a,z,t) = 0, \quad 0 < z < L, \quad t > 0,$$
$$U(r,z,0) = (a^2 - r^2)(L - z), \quad 0 < r < a, \quad 0 < z < L,$$

(a) by using the results of Exercise 19, Example 9.1, and Exercise 1(a) in Section 6.2.
(b) by separation of variables.

Part B Vibrations

21. (a) A vibrating circular membrane of radius a is given an initial displacement that is a function only of r, namely, $f(r)$, $0 \le r \le a$, and zero initial velocity. Show that subsequent displacements of the membrane, if its edge $r = a$ is fixed on the xy-plane, are of the form

$$z(r,t) = \frac{\sqrt{2}}{a}\sum_{n=1}^{\infty} A_n \cos c\lambda_n t \frac{J_0(\lambda_n r)}{J_1(\lambda_n a)}.$$

What is A_n?

(b) The first term in the series in part (a), called the **fundamental mode of vibration** for the membrane, is

$$H_1(r,t) = \frac{\sqrt{2}}{a} A_1 \cos c\lambda_1 t \frac{J_0(\lambda_1 r)}{J_1(\lambda_1 a)}.$$

Simplify and describe this mode when $a = 1$. Does $H_1(r,t)$ have nodal curves?

(c) Repeat part (b) for the second mode of vibration.

(d) Are frequencies of higher modes of vibration integer multiples of the frequency of the fundamental mode? Were they for a vibrating string with fixed ends?

22. A circular membrane of radius a has its edge fixed on the xy-plane. In addition, a clamp holds the membrane on the xy-plane along a radial line from the centre to the circumference. If the membrane is released from rest at a displacement $f(r,\theta)$, find subsequent displacements. (For consistency, we require $f(r,\theta)$ to vanish along the clamped radial line.)

23. Simplify the solution in part (a) of Exercise 21 when $f(r) = a^2 - r^2$. (See Example 9.1.)

24. A circular membrane of radius a is parallel to the xy-plane and is falling with constant speed v_0. At time $t = 0$, it strikes the xy-plane. For $t > 0$, the edge of the membrane is fixed on the

xy-plane, but the remainder of the membrane is free to vibrate vertically. Find displacements of the membrane.

25. Equation 9.29 with coefficients defined in 9.31 describes displacements of a circular membrane with fixed edge when oscillations are initiated from rest at some prescribed displacement. In this exercise we examine nodal curves for various modes of vibration.

(a) The first mode of vibration is the term $(d_{01}/\sqrt{2\pi})R_{01}(r)\cos c\lambda_{01}t$. Show that this mode has no nodal curves.

(b) Show that the mode $(d_{02}/\sqrt{2\pi})R_{02}(r)\cos c\lambda_{02}t$ has one nodal curve, a circle.

(c) Show that the mode $(d_{03}/\sqrt{2\pi})R_{03}(r)\cos c\lambda_{03}t$ has two circular nodal curves.

(d) On the basis of parts (a), (b), and (c), what are the nodal curves for the mode $(d_{0n}/\sqrt{2\pi})R_{0n}(r)\cos c\lambda_{0n}t$?

(e) Corresponding to $n = m = 1$, there are two modes, $(d_{11}/\sqrt{\pi})R_{11}(r)\cos c\lambda_{11}t\cos\theta$ and $(f_{11}/\sqrt{\pi})R_{11}(r)\cos c\lambda_{11}t\sin\theta$. Show that each of these modes has only one nodal curve, a straight line.

(f) Find nodal curves for the modes $(d_{12}/\sqrt{\pi})R_{12}(r)\cos c\lambda_{12}t\cos\theta$ and $(f_{12}/\sqrt{\pi})R_{12}(r)\cos c\lambda_{12}t\sin\theta$.

(g) Find nodal curves for the modes $(d_{22}/\sqrt{\pi})R_{22}(r)\cos c\lambda_{22}t\cos 2\theta$ and $(f_{22}/\sqrt{\pi})R_{22}(r)\cos c\lambda_{22}t\sin 2\theta$.

(h) On the basis of parts (e), (f), and (g), what are the nodal curves for the modes $(d_{mn}/\sqrt{\pi})R_{mn}(r)\cos c\lambda_{mn}t\cos m\theta$ and $(f_{mn}/\sqrt{\pi})R_{mn}(r)\cos c\lambda_{mn}t\sin m\theta$?

26. The initial boundary value problem for small horizontal displacements of a suspended cable when gravity is the only force acting on the cable is

$$\begin{aligned}
\frac{\partial^2 y}{\partial t^2} &= g\frac{\partial}{\partial x}\left(x\frac{\partial y}{\partial x}\right), \quad 0 < x < L, \quad t > 0,\\
y(L,t) &= 0, \quad t > 0,\\
y(x,0) &= f(x), \quad 0 < x < L,\\
y_t(x,0) &= h(x), \quad 0 < x < L.
\end{aligned}$$

(See Exercise 26 in Section 2.3.)

(a) Show that when a new independent variable $z = \sqrt{4x/g}$ is introduced, $y(z,t)$ must satisfy

$$\begin{aligned}
\frac{\partial^2 y}{\partial t^2} &= \frac{1}{z}\frac{\partial}{\partial z}\left(z\frac{\partial y}{\partial z}\right), \quad 0 < z < M, \quad t > 0,\\
y(M,t) &= 0, \quad t > 0,\\
y(z,0) &= f(gz^2/4), \quad 0 < z < M,\\
y_t(z,0) &= h(gz^2/4), \quad 0 < z < M,
\end{aligned}$$

where $M = \sqrt{4L/g}$.

(b) Solve this problem by separation of variables, and hence find $y(x,t)$.

27. Multidimensional eigenfunctions for problem 9.20 are solutions of the two-dimensional eigenvalue problem

$$\begin{aligned}
\frac{\partial^2 W}{\partial r^2} + \frac{1}{r}\frac{\partial W}{\partial r} + \frac{1}{r^2}\frac{\partial^2 W}{\partial \theta^2} + \lambda^2 W &= 0, \quad 0 < r < a, \quad -\pi < \theta \le \pi,\\
W(a,\theta) &= 0, \quad -\pi < \theta \le \pi.
\end{aligned}$$

(a) Find eigenfunctions (normalized with respect to the unit weight function over the circle $r \le a$).
(b) Use the eigenfunctions in part (a) to solve problem 9.20.

Part C Potential, Steady-state Heat Conduction, Static Deflections of Membranes

28. (a) Solve the following boundary value problem associated with the Helmholtz equation on a circle

$$\begin{aligned} \nabla^2 V + k^2 V = 0, &\quad 0 < r < a, \quad -\pi < \theta \le \pi \quad (k > 0 \text{ a constant}) \\ V(a,\theta) = f(\theta), &\quad -\pi < \theta \le \pi. \end{aligned}$$

(b) Is $V(0,\theta)$ the average value of $f(\theta)$ on $r = a$?
(c) What is the solution when $f(\theta) = 1$?

29. Solve the following problem for potential in a cylinder

$$\begin{aligned} \frac{\partial^2 V}{\partial r^2} + \frac{1}{r}\frac{\partial V}{\partial r} + \frac{\partial^2 V}{\partial z^2} = 0, &\quad 0 < r < a, \quad 0 < z < L, \\ V(a,z) = 0, &\quad 0 < z < L, \\ V(r,0) = 0, &\quad 0 < r < a, \\ V(r,L) = f(r), &\quad 0 < r < a. \end{aligned}$$

30. Find the potential inside a cylinder of length L and radius a when potential on the curved surface is zero and potentials on the flat ends are nonzero.

31. (a) Find the steady-state temperature in a cylinder of radius a and length L if the end $z = 0$ is maintained at temperature $f(r)$, the end $z = L$ is kept at temperature zero, and heat is transferred on $r = a$ to a medium at temperature zero according to Newton's law of cooling.
(b) Simplify the solution when $f(r) = U_0$, a constant.

32. The temperature in a semi-infinite cylinder $0 < r < a$, $z > 0$ is in a steady-state situation. Find the temperature if the cylindrical wall is at temperature zero and the temperature of the base $z = 0$ is $f(r)$.

33. Repeat Exercise 32 if the cylindrical wall is insulated.

34. Use separation of variables to find the potential inside a sphere of radius a when the potential on the sphere is a function $f(\phi)$ of ϕ only. Does the solution for Example 9.4 specialize to this result? What is the solution when $f(\phi)$ is a constant function?

35. Show that if the potential on the surface of a sphere is a function $f(\theta)$ of θ only, the potential interior to the sphere is still a function of r, ϕ, and θ.

36. Find the potential interior to a sphere of radius a when the potential must satisfy a Neumann condition on the sphere,

$$\frac{\partial V(a,\phi,\theta)}{\partial r} = f(\phi,\theta), \quad 0 \le \phi \le \pi, \quad -\pi < \theta \le \pi.$$

37. Find the potential interior to a sphere of radius a when the potential must satisfy a Robin condition on the sphere,

$$l\frac{\partial V(a,\phi,\theta)}{\partial r} + hV(a,\phi,\theta) = f(\phi,\theta), \quad 0 \le \phi \le \pi, \quad -\pi < \theta \le \pi.$$

38. Find the steady-state temperature inside a hemisphere $r \le a$, $z \ge 0$ when temperature on $z = 0$ is zero and that on $r = a$ is a function of ϕ only. (Hint: See Exercise 5 in Section 8.6.) Simplify the solution when $f(\phi)$ is a constant function.

39. Repeat Exercise 38 if the base of the hemisphere is insulated. (Hint: See Exercise 6 in Section 8.6.)

40. Find the bounded potential outside the hemisphere $r \le a$, $z \ge 0$ when potential on $z = 0$ is zero and that on $r = a$ is a function of ϕ only. (Hint: See the results of Exercise 5 in Section 8.6.)

41. Find the potential interior to a sphere of radius a when the potential on the upper half is a constant V_0 and the potential on the lower half is zero.

42. Use the result of Exercise 41 to find the potential inside a sphere of radius a when potentials on the top and bottom halves are constant values V_0 and V_1, respectively.

43. Find the potential in the region between two concentric spheres when the potential on each sphere is
(a) a constant;
(b) a function of ϕ only (and show that the solution reduces to that in part (a) when the functions are constant;
(c) a function of ϕ and θ (and show that the solution reduces to that in part (b) when the functions depend only on ϕ.

44. (a) Show that the negative of Poisson's integral formula 9.41 is the solution to Laplace's equation exterior to the sphere $r = a$ if $V(r,\phi,\theta)$ is required to vanish at infinity.
(b) Show that if $V(r,\phi,\theta)$ is the solution to the interior problem, then $(a/r)V(a^2/r,\phi,\theta)$ is the solution to the exterior problem. Do this using the result in part (a), and also by checking that the function satisfies the boundary value problem.

45. (a) What is the potential interior to a sphere of radius a when its value on the sphere is a constant V_0?
(b) Determine the potential exterior to a sphere of radius a when its value on the sphere is a constant V_0, and the potential must vanish at infinity. Do this in two ways, using separation of variables, and the result of Exercise 44.

46. What is the potential exterior to a sphere of radius a when the potential must vanish at infinity and satisfy a Neumann condition on the sphere,

$$-\frac{\partial V(a,\phi,\theta)}{\partial r} = f(\phi,\theta), \quad 0 \le \phi \le \pi, \quad -\pi < \theta \le \pi.$$

47. What is the potential exterior to a sphere of radius a when the potential must vanish at infinity and satisfy a Robin condition on the sphere,

$$-l\frac{\partial V(a,\phi,\theta)}{\partial r} + hV(a,\phi,\theta) = f(\phi,\theta), \quad 0 \le \phi \le \pi, \quad -\pi < \theta \le \pi.$$

48. Consider the following boundary value problem for steady-state temperature inside a cylinder of length L and radius a when the temperature of each end is zero:

$$\frac{\partial^2 U}{\partial r^2} + \frac{1}{r}\frac{\partial U}{\partial r} + \frac{\partial^2 U}{\partial z^2} = 0, \quad 0 < r < a, \quad 0 < z < L,$$
$$U(r, 0) = 0, \quad 0 < r < a,$$
$$U(r, L) = 0, \quad 0 < r < a,$$
$$U(a, z) = f(z), \quad 0 < z < L.$$

(a) Verify that separation of variables $U(r, z) = R(r)Z(z)$ leads to a Sturm-Liouville system in $Z(z)$ and the following differential equation in $R(r)$:

$$r\frac{d^2R}{dr^2} + \frac{dR}{dr} - \lambda^2 rR = 0, \quad 0 < r < a.$$

(b) Show that the change of variable $x = \lambda r$ leads to Bessel's modified differential equation of order zero,

$$x\frac{d^2R}{dx^2} + \frac{dR}{dx} - xR = 0.$$

(See Exercise 10 in Section 8.3.)

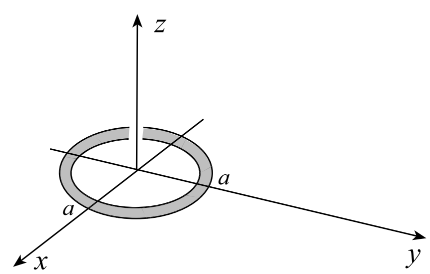
(c) Find functions $R_n(r)$ corresponding to eigenvalues λ_n, and use superposition to solve the boundary value problem.

(d) Simplify the solution in part (c) in the case that $f(z)$ is a constant value U_0.

49. Solve the boundary value problem in Exercise 48 if the ends of the cylinder are insulated.

50. (a) A charge Q is distributed uniformly around a thin ring of radius a in the xy-plane with centre at the origin (figure to the right). Show that potential at every point on the z-axis due to this charge is

$$V = \frac{Q}{4\pi\epsilon_0\sqrt{a^2 + r^2}}.$$



(b) The potential at other points in space must be independent of the spherical coordinate θ. Show that $V(r, \phi)$ must be of the form

$$V(r, \phi) = \sum_{n=0}^{\infty} \left(A_n r^n + \frac{B_n}{r^{n+1}}\right)\sqrt{\frac{2n+1}{2}}P_n(\cos\phi).$$

What does this result predict for potential at points on the positive z-axis?

(c) Equate expressions from parts (a) and (b) for V on the positive z-axis and expand $1/\sqrt{a^2 + r^2}$ in powers of r/a and a/r to find $V(r, \phi)$.

51. Repeat Exercise 50 in the case that charge Q is distributed uniformly over a disc of radius a in the xy-plane with centre at the origin.