§15.5 Method of Weighted Residuals and Dirichlet Boundary Value Problems

We now apply the MWR to boundary value problems associated with partial differential equations, and in this section, we deal with Dirichlet problems as they are the easiest to handle. We begin with a general discussion to outline one possible procedure, but other approaches may be advantageous, such as *reduction of dimensionality*, a method that we also introduce in this section. Consider the two-dimensional problem

$$L(V) = F(x,y), \quad (x,y) \text{ in } R, \tag{15.41a}$$

$$V(x,y) = G(x,y), \quad (x,y) \text{ on } \beta(R), \tag{15.41b}$$

where L is some partial differential operator (which may be linear, such as the Laplacian, or nonlinear), $\beta(R)$ is the boundary of some region R in the xy-plane, and $F(x,y)$ and $G(x,y)$ are given functions. For an interior method, we could take approximations in the form

$$V_N(x,y) = \phi_0(x,y) + \sum_{n=1}^{N} c_n\phi_n(x,y), \tag{15.42}$$

where $\phi_0(x,y)$ satisfies the nonhomogeneous boundary condition, and basis functions $\phi_n(x,y)$ for $n = 1,\ldots,N$ satisfy the homogeneous version of the boundary condition; that is $\phi_n(x,y) = 0$ on $\beta(R)$. Approximations $V_N(x,y)$ then satisfy boundary condition 15.41b, and the resulting (equation) residual need only account for $V_N(x,y)$ not satisfying the PDE,

$$\begin{aligned} R = L(V_N) - F(x,y) &= L\left[\phi_0(x,y) + \sum_{n=1}^{N} c_n\phi_n(x,y)\right] - F(x,y) \\ &= L(\phi_0) + \sum_{n=1}^{N} c_n L(\phi_n) - F(x,y). \end{aligned} \tag{15.43}$$

(This calculation has assumed that L is linear.) When N weight functions $w_m(x,y)$ are chosen, the MWR requires

$$0 = \iint_R \left[L(\phi_0) + \sum_{n=1}^{N} c_n L(\phi_n) - F(x,y)\right] w_m(x,y)\,dA, \quad m = 1,\ldots,N.$$

We can express these equations, which determine the c_n, in the form

$$\sum_{n=1}^{N} c_n \iint_R L(\phi_n) w_m\,dA = \iint_R [F - L(\phi_0)] w_m\,dA, \quad m = 1,\ldots,N.$$

As was the case for boundary value problems associated with ODEs, basis functions must be linearly independent and from a complete set. Possible choices once again include eigenfunctions of associated Sturm-Liouville problems, and polynomials. The polynomials $x^n y^m$, $n, m = 0, 1, 2, \ldots$ form a complete set for the space of continuous functions, but in the above approach, it is unlikely that they will satisfy the boundary condition. However, if $\omega(x,y)$ is a positive, continuously differentiable function in R that vanishes on the boundary of R, then the functions $\omega(x,y)x^n y^m$

constitute a complete set and they do satisfy the boundary condition. We use this idea in our first example which has a simple nonhomogeneity in the differential equation, and homogeneous boundary conditions.

Example 15.2 Find polynomial approximations to the solution of the boundary value problem

$$\begin{aligned}\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} &= k, \quad -L < x < L, \quad -L < y < L,\\ V(-L,y) = V(L,y) &= 0, \quad -L < y < L,\\ V(x,-L) = V(x,L) &= 0, \quad -L < x < L,\end{aligned}$$

where k is a constant. Use an interior method and a boundary method. To four decimal places, the solution of the boundary value problem at the centre of the square is $-0.2947L^2k$ (see Exercise 32 in Section 4.3). Compare this value to that predicted by each approximation.

Solution Interior Method

The function $\omega(x,y) = (L^2 - x^2)(L^2 - y^2)$ is positive, continuously differentiable, and vanishes on the edges of the square. Polynomial basis functions can therefore be taken as $x^n y^m (L^2 - x^2)(L^2 - y^2)$, $n, m = 0, 1, \ldots$. Furthermore, because the solution of the problem should be an even function of both x and y, and be symmetric in x and y, we can further restrict the choices for $x^n y^m$. First and second approximations that satisfy the boundary conditions and symmetry requirements are $V_1(x,y) = c(L^2 - x^2)(L^2 - y^2)$ and $V_2(x,y) = (L^2 - x^2)(L^2 - y^2)[c + d(x^2 + y^2)]$. We work with the second approximation. The equation residual is

$$\begin{aligned}R(x,y) &= -2(L^2 - y^2)[c + d(x^2 + y^2)] + 2(-2x)(L^2 - y^2)(2dx)\\ &\quad + (L^2 - x^2)(L^2 - y^2)(2d) - 2(L^2 - x^2)[c + d(x^2 + y^2)]\\ &\quad + 2(-2y)(L^2 - x^2)(2dy) + (L^2 - x^2)(L^2 - y^2)(2d) - k\\ &= -2[c + d(x^2 + y^2)](2L^2 - x^2 - y^2) - 8d[x^2(L^2 - y^2) + y^2(L^2 - x^2)]\\ &\quad + 4d(L^2 - x^2)(L^2 - y^2) - k.\end{aligned}$$

We use collocation, subdomains, and Galerkin's method to find values for c and d.

Collocation

Due to the symmetry of the problem, we choose two collocation points in the first quadrant part of the square, namely $(0,0)$ and $(L/2, L/2)$. These yield the equations

$$0 = -4cL^2 + 4dL^4 - k, \qquad 0 = -3cL^2 - \frac{9}{4}dL^4 - k.$$

The solution is $c = -\dfrac{25k}{84L^2}$ and $d = -\dfrac{k}{21L^4}$, and therefore the second collocation approximation is

$$V_2(x,y) = -\frac{k}{84L^4}(L^2 - x^2)(L^2 - y^2)[25L^2 + 4(x^2 + y^2)].$$

It predicts a value of $V_2(0,0) = -25L^2k/84 \approx -0.2976L^2k$ for the centre of the region.

Subdomain

We choose two symmetric subdomains, namely the square $A_1 : 0 \leq x, y \leq L/2$ and A_2 as the remainder of the original square in the first quadrant. These require

$$0 = \int_0^{L/2} \int_0^{L/2} R(x,y)\, dy\, dx = -\frac{kL^2}{4} - \frac{11cL^4}{12} + \frac{31dL^6}{80},$$
$$0 = \int_0^{L/2} \int_{L/2}^{L} R(x,y)\, dy\, dx + \int_{L/2}^{L} \int_0^{L} R(x,y)\, dy\, dx = -\frac{3kL^2}{4} - \frac{7cL^4}{4} - \frac{287dL^6}{80}.$$

The solution of these equations is $c = -\dfrac{285k}{952L^2}$ and $d = -\dfrac{15k}{238L^4}$, and the second subdomain approximation is

$$V_2(x,y) = -\frac{15k}{952L^4}(L^2 - x^2)(L^2 - y^2)[17L^2 - 4(x^2 + y^2)].$$

It predicts a value of $V_2(0,0) = -285kL^2/952 \approx -0.2994L^2k$ for the centre of the region.

Galerkin

Galerkin's method requires

$$0 = \int_0^L \int_0^L R(x,y)(L^2 - x^2)(L^2 - y^2)\, dy\, dx = -\frac{4kL^6}{9} - \frac{64cL^8}{45} - \frac{256dL^{10}}{525},$$
$$0 = \int_0^L \int_0^L R(x,y)(L^2 - x^2)(L^2 - y^2)(x^2 + y^2)\, dy\, dx = -\frac{8kL^8}{45} - \frac{256cL^{10}}{525} - \frac{2816dL^{12}}{4725}.$$

The solution of these equations is $c = -\dfrac{1295k}{4432L^2}$ and $d = -\dfrac{525k}{8864L^4}$. The second Galerkin approximation is therefore

$$V_2(x) = -\frac{5k}{8864L^2}(L^2 - x^2)(L^2 - y^2)\left[518L^2 + 105(x^2 + y^2)\right].$$

It predicts a value of $V_2(0,0) = -1295kL^2/4432 \approx -0.2922L^2k$ for the centre of the region.

Boundary Method

In a boundary method, polynomial approximations must satisfy the PDE. The function $k(x^2 + y^2)/4$ satisfies the PDE. To find polynomials that satisfy the homogeneous version of the PDE, namely, Laplace's equation, we use the fact that real and imaginary parts of every complex analytic function satisfy Laplace's equation; in particular, real and imaginary parts of the function $z^n = (x + yi)^n$ give polynomial solutions of Laplace's equation. The first few, from $n = 1, 2, 3$, and 4 are

$$1,\ x,\ y,\ x^2 - y^2,\ xy,\ x^3 - 3xy^2,\ 3x^2y - y^3,\ x^4 - 6x^2y^2 + y^4,\ 4x^3 - 4xy^3, \ldots.$$

As already noted, the solution of the problem must be even in x and y, and symmetric with respect to x and y. The first two such polynomials are 1 and $x^4 - 6x^2y^2 + y^4$. We therefore take as an approximating polynomial that satisfies the PDE

$$V_2(x,y) = c + \frac{k}{4}(x^2 + y^2) + d(x^4 - 6x^2y^2 + y^4).$$

The residual of this approximation along each of the four edges of the square is identical, and we therefore consider it along $x = L$,

$$R = c + \frac{k}{4}(L^2 + y^2) + d(L^4 - 6L^2y^2 + y^4).$$

We now use collocation, subdomains, and Galerkin's method to determine c and d.

Collocation

Collocation with $y = L/3$ and $y = 2L/3$ requires

$$0 = c + \frac{k}{4}\left(L^2 + \frac{L^2}{9}\right) + d\left(L^4 - \frac{2L^4}{3} + \frac{L^4}{81}\right),$$
$$0 = c + \frac{k}{4}\left(L^2 + \frac{4L^2}{9}\right) + d\left(L^4 - \frac{8L^4}{3} + \frac{16L^4}{81}\right).$$

These imply that $c = -37kL^2/126$ and $d = 9k/(196L^2)$, and the second collocation approximation is

$$V_2(x, y) = -\frac{37kL^2}{126} + \frac{k}{4}(x^2 + y^2) + \frac{9k}{196L^2}(x^4 - 6x^2y^2 + y^4).$$

It predicts a value of $-37kL^2/126 \approx -0.2937L^2k$ at $(0, 0)$.

Subdomain

Subdomains require

$$0 = \int_0^{L/2}\left[c + \frac{k}{4}(L^2 + y^2) + d(L^4 - 6L^2y^2 + y^4)\right] dy = \frac{Lc}{2} + \frac{13L^3k}{96} + \frac{41L^5d}{160},$$
$$0 = \int_{L/2}^{L}\left[c + \frac{k}{4}(L^2 + y^2) + d(L^4 - 6L^2y^2 + y^4)\right] dy = \frac{Lc}{2} + \frac{19L^3k}{96} - \frac{169L^5d}{160}.$$

These give $c = -31kL^2/105$ and $d = k/(21L^2)$, and the second subdomain approximation is

$$V_2(x, y) = -\frac{31kL^2}{105} + \frac{k}{4}(x^2 + y^2) + \frac{k}{21L^2}(x^4 - 6x^2y^2 + y^4).$$

It predicts $V_2(0, 0) = -31kL^2/105 \approx -0.2952L^2k$.

Galerkin

Galerkin's method requires

$$0 = \int_0^{L}\left[c + \frac{k}{4}(L^2 + y^2) + d(L^4 - 6L^2y^2 + y^4)\right] dy = Lc + \frac{kL^3}{3} - \frac{4L^5d}{5},$$
$$0 = \int_0^{L}\left[c + \frac{k}{4}(L^2 + y^2) + d(L^4 - 6L^2y^2 + y^4)\right](L^4 - 6L^2y^2 + y^4)\, dy$$
$$= -\frac{4L^5}{5} - \frac{8L^7k}{21} + \frac{944L^9d}{315}.$$

The solution is $c = -205kL^2/696$ and $d = 45k/(928L^2)$, and the second Galerkin approximation is

$$V_2(x, y) = -\frac{205kL^2}{696} + \frac{k}{4}(x^2 + y^2) + \frac{45k}{928L^2}(x^4 - 6x^2y^2 + y^4).$$

Its prediction at the centre is $-205kL^2/696 \approx -0.2945L^2k$.•

The next example has a more general nonhomogeneity in the differential equation and two nonhomogeneous boundary conditions. We also use it to introduce the method of *reduction of dimensionality.*

Example 15.3 Use polynomials and eigenfunctions to approximate the solution of the boundary value problem

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = F(x,y), \quad 0 < x < L, \quad 0 < y < L', \tag{15.44a}$$

$$V(0,y) = V(L,y) = 0, \quad 0 < y < L', \tag{15.44b}$$

$$V(x,0) = g(x), \quad 0 < x < L, \tag{15.44c}$$

$$V(x,L') = h(x), \quad 0 < x < L. \tag{15.44d}$$

For continuity of boundary conditions at the corners of the rectangle, assume that nonhomogeneities $g(x)$ and $h(x)$ satisfy the conditions $g(0) = g(L) = h(0) = h(L) = 0$.

Solution Consider using approximations of the form

$$V_N(x,y) = \phi_0(x,y) + \sum_{n=1}^{N} c_n \phi_n(x,y), \tag{15.45}$$

where $\phi_0(x,y)$ satisfies all boundary conditions, homogeneous and nonhomogeneous, and basis function $\phi_n(x,y)$ for $n = 1, \ldots, N$ satisfy homogeneous versions of the boundary conditions. A convenient choice for $\phi_0(x,y)$ is $g(x)(1 - y/L') + h(x)y/L'$. (Can you see the difficulty at this point were $g(0)$, $g(L)$, $h(0)$, and/or $h(L)$ nonzero?) For polynomial approximations, we choose $\omega(x,y) = xy(L-x)(L'-y)$, in which case basis functions that satisfy homogeneous versions of the boundary conditions are $\phi_{nm}(x,y) = x^n y^m (L-x)(L'-y)$, $n, m = 1, 2, \ldots$. Approximate solutions of the problem are therefore

$$V_N(x,y) = g(x)\left(1 - \frac{y}{L'}\right) + h(x)\frac{y}{L'} + \sum_{n=1}^{N}\sum_{m=1}^{N} c_{nm} x^n y^m (L-x)(L'-y). \tag{15.46}$$

The first approximation is

$$V_1(x,y) = g(x)\left(1 - \frac{y}{L'}\right) + h(x)\frac{y}{L'} + c_{11}xy(L-x)(L'-y), \tag{15.47}$$

with equation residual

$$R = g''(x)\left(1 - \frac{y}{L'}\right) + h''(x)\frac{y}{L'} + 2c_{11}(x^2 - Lx + y^2 - L'y) - F(x,y).$$

Galerkin's method requires

$$0 = \int_0^L \int_0^{L'} \left[g''(x)\left(1 - \frac{y}{L'}\right) + h''(x)\frac{y}{L'} + 2c_{11}(x^2 - Lx + y^2 - L'y) \right.$$
$$\left. - F(x,y)\right] xy(L-x)(L'-y)\, dy\, dx,$$

and integrations lead to

$$c_{11} = \frac{-90}{L^3 L'^3(L^2 + L'^2)} \left[\frac{L'^3}{6} \int_0^L g(x)\,dx + \frac{L'^3}{6} \int_0^L h(x)\,dx + \int_0^L \int_0^{L'} F(x,y) xy(L-x)(L'-y)\,dy\,dx \right].$$

We now consider using eigenfunctions of the associated eigenvalue problem

$$\phi_{nm}(x,y) = \sin\frac{n\pi x}{L} \sin\frac{m\pi y}{L'}$$

as basis functions. Approximations are

$$V_{NM}(x,y) = \sum_{n=1}^{N} \sum_{m=1}^{M} c_{nm} \sin\frac{n\pi x}{L} \sin\frac{m\pi y}{L'}. \tag{15.48}$$

These approximations satisfy homogeneous boundary conditions 15.44b, but not nonhomogeneous conditions 15.44c,d. The solution is pursued when all boundary conditions are homogeneous in Exercise 5. Nonhomogeneous conditions 15.44c,d can be handled by transforming them into the PDE. Suppose we make a change of dependent variable by $U(x,y) = V(x,y) + \phi_0(x,y)$ where $\phi_0(x,y)$ is any function that has value $g(x)$ along $y = 0$ and $h(x)$ along $y = L'$. The obvious choice is $g(x)(1 - y/L') + h(x)y/L'$. With this change, the boundary value problem for $U(x,y)$ is

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = F(x,y) - g''(x)\left(1 - \frac{y}{L'}\right) - h''(x)\frac{y}{L'}, \quad 0 < x < L, \quad 0 < y < L',$$
$$U(0,y) = U(L,y) = 0, \quad 0 < y < L',$$
$$U(x,0) = U(x,L') = 0, \quad 0 < x < L.$$

Approximations

$$U_{NM}(x,y) = \sum_{n=1}^{N} \sum_{m=1}^{M} c_{nm} \sin\frac{n\pi x}{L} \sin\frac{m\pi y}{L'}$$

satisfy the homogeneous boundary conditions. The equation residual is

$$R = -\pi^2 \sum_{n=1}^{N} \sum_{m=1}^{M} c_{nm} \left(\frac{n^2}{L^2} + \frac{m^2}{L'^2}\right) \sin\frac{n\pi x}{L} \sin\frac{m\pi y}{L'} - F(x,y) + g''(x)\left(1 - \frac{y}{L'}\right) + h''(x)\frac{y}{L'}.$$

Galerkin's method requires

$$0 = \int_0^L \int_0^{L'} \left[-\pi^2 \sum_{n=1}^{N} \sum_{m=1}^{M} c_{nm} \left(\frac{n^2}{L^2} + \frac{m^2}{L'^2}\right) \sin\frac{k\pi x}{L} \sin\frac{l\pi y}{L'} - F(x,y) + g''(x)\left(1 - \frac{y}{L'}\right) + h''(x)\frac{y}{L'} \right] \sin\frac{k\pi x}{L} \sin\frac{l\pi y}{L'} dy\,dx.$$

Due to the orthogonality of the eigenfunctions, this immediately leads to

$$c_{nm} = \frac{-4LL'}{\pi^2(n^2L'^2 + m^2L^2)} \int_0^L \int_0^{L'} \left[F(x,y) - g''(x)\left(1 - \frac{y}{L'}\right) - h''(x)\frac{y}{L'}\right] * \sin\frac{n\pi x}{L}\sin\frac{m\pi y}{L'}dy\,dx. \tag{15.49a}$$

Multiple integrations by parts on the terms involving $g(x)$ and $h(x)$ leads to the alternative formula

$$c_{nm} = \frac{-4LL'}{\pi^2(n^2L'^2 + m^2L^2)}\left\{\int_0^L \int_0^{L'} F(x,y)\sin\frac{n\pi x}{L}\sin\frac{m\pi y}{L'}dy\,dx + \frac{n^2\pi L'}{mL^2}\int_0^L [g(x) + (-1)^{m+1}h(x)]\sin\frac{n\pi x}{L}dx\right\}. \tag{15.49b}$$

Finally then

$$V_{NM}(x,y) = \sum_{n=1}^{N}\sum_{m=1}^{M} c_{nm}\sin\frac{n\pi x}{L}\sin\frac{m\pi y}{L'} - g(x)\left(1 - \frac{y}{L'}\right) - h(x)\frac{y}{L'}.\bullet$$

Reduction of Dimensionality

The MWR can be used to reduce the dimensionality of a problem; for problem 15.44, the PDE is reduced to an ODE. We represent approximations as sums of separated functions

$$V_N(x,y) = \sum_{n=1}^{N} \phi_n(x,y) = \sum_{n=1}^{N} c_n(y)\psi_n(x),$$

where basis functions $\psi_n(x)$ must be specified, and coefficients $c_n(y)$ will be determined by the MWR. (Approximations were separated in the previous approach, but they need not have been so.) According to Table 15.5, polynomial basis functions satisfying boundary conditions 15.44b are $\psi_n(x) = x^n(L - x)$, $n = 1, 2, \ldots$. We therefore take approximations in the form

$$V_N(x,y) = \sum_{n=1}^{N} c_n(y)x^n(L - x),$$

the first being

$$V_1(x,y) = c_1(y)x(L - x).$$

The equation residual is

$$R = -2c_1 + x(L - x)c_1'' - F(x,y).$$

Galerkin's method requires

$$0 = \int_0^L [-2c_1 + x(L - x)c_1'' - F(x,y)]x(L - x)\,dx.$$

Integrations lead to

$$c_1'' - \frac{10c_1}{L^2} = \frac{30}{L^5}\int_0^L F(x,y)x(L-x)\,dx.$$

So that the remainder of the procedure can be illustrated without unduly complicated calculations, we assume that $F(x,y) = k$, a constant. In this case, $c_1(y)$ must satisfy the ODE

$$c_1'' - \frac{10c_1}{L^2} = \frac{5k}{L^2}.$$

A general solution of this equation is

$$c_1(y) = A\cosh\frac{\sqrt{10}y}{L} + B\sinh\frac{\sqrt{10}y}{L} - \frac{k}{2}.$$

The first approximation is therefore

$$V_1(x,y) = \left[A\cosh\frac{\sqrt{10}y}{L} + B\sinh\frac{\sqrt{10}y}{L} - \frac{k}{2}\right]x(L-x).$$

We associate boundary residuals with this approximation due to the fact that it does not satisfy boundary conditions 15.44c,d,

$$R_{|y=0} = \left(A - \frac{k}{2}\right)x(L-x) - g(x),$$
$$R_{|y=L'} = \left[A\cosh\frac{\sqrt{10}L'}{L} + B\sinh\frac{\sqrt{10}L'}{L} - \frac{k}{2}\right]x(L-x) - h(x).$$

We apply Galerkin's method to find A and B,

$$0 = \int_0^L \left[\left(A - \frac{k}{2}\right)x(L-x) - g(x)\right]x(L-x)\,dx,$$
$$0 = \int_0^L \left\{\left[A\cosh\frac{\sqrt{10}L'}{L} + B\sinh\frac{\sqrt{10}L'}{L} - \frac{k}{2}\right]x(L-x) - h(x)\right\}x(L-x)\,dx.$$

Integrations lead to

$$A = \frac{k}{2} + \frac{30}{L^5}\int_0^L g(x)x(L-x)\,dx,$$
$$B = \frac{30}{L^5}\operatorname{csch}\frac{\sqrt{10}L'}{L}\int_0^L h(x)x(L-x)\,dx - \coth\frac{\sqrt{10}L'}{L}\left[\frac{k}{2} + \frac{30}{L^5}\int_0^L g(x)x(L-x)\,dx\right].$$

We can also reduce the dimensionality of the problem using eigenfunctions $\psi_n(x) = \sin(n\pi x/L)$ as basis functions,

$$V_N(x,y) = \sum_{n=1}^N c_n(y)\sin\frac{n\pi x}{L}.$$

The equation residual is

$$R = \sum_{n=1}^{N} \left(-\frac{n^2\pi^2}{L^2}\right) c_n \sin\frac{i\pi x}{L} + \sum_{n=1}^{N} c_n'' \sin\frac{n\pi x}{L} - F(x,y)$$
$$= \sum_{n=1}^{N} \left(c_n'' - \frac{n^2\pi^2}{L^2} c_n\right) \sin\frac{n\pi x}{L} - F(x,y).$$

Galerkin's method requires

$$0 = \int_0^L \left[\sum_{n=1}^{N} \left(c_n'' - \frac{n^2\pi^2}{L^2} c_n\right) \sin\frac{n\pi x}{L} - F(x,y)\right] \sin\frac{m\pi x}{L}\, dx,$$

and due to orthogonality of eigenfunctions, this reduces to

$$c_m'' - \frac{m^2\pi^2}{L^2} c_m = \frac{2}{L}\int_0^L F(x,y) \sin\frac{m\pi x}{L}\, dx.$$

When $F(x,y) = k$, a constant, integration gives

$$c_m'' - \frac{m^2\pi^2}{L^2} c_m = \frac{2}{L}\int_0^L k \sin\frac{m\pi x}{L}\, dx = \frac{2k[1 + (-1)^{m+1}]}{m\pi}.$$

A general solution of this ODE is

$$c_m(y) = A_m \cosh\frac{m\pi y}{L} + B_m \sinh\frac{m\pi y}{L} - \frac{2kL^2[1 + (-1)^{m+1}]}{m^3\pi^3},$$

and the N^{th} approximation is

$$V_N(x,y) = \sum_{n=1}^{N} \left[A_n \cosh\frac{n\pi y}{L} + B_n \sinh\frac{n\pi y}{L} - \frac{2kL^2[1 + (-1)^{n+1}]}{n^3\pi^3}\right] \sin\frac{n\pi x}{L}.$$

To evaluate A_n and B_n, we form boundary residuals along $y = 0$ and $y = L'$,

$$R_{|y=0} = \sum_{n=1}^{N} \left[A_n - \frac{2kL^2[1 + (-1)^{n+1}]}{n^3\pi^3}\right] \sin\frac{n\pi x}{L} - g(x),$$
$$R_{|y=L'} = \sum_{n=1}^{N} \left[A_n \cosh\frac{n\pi L'}{L} + B_n \sinh\frac{n\pi L'}{L} - \frac{2kL^2[1 + (-1)^{n+1}]}{n^3\pi^3}\right] \sin\frac{n\pi x}{L} - h(x).$$

Application of Galerkin's method gives

$$0 = \int_0^L \left\{\sum_{n=1}^{N} \left[A_n - \frac{2kL^2[1 + (-1)^{n+1}]}{n^3\pi^3}\right] \sin\frac{n\pi x}{L} - g(x)\right\} \sin\frac{m\pi x}{L}\, dx,$$
$$0 = \int_0^L \left\{\sum_{n=1}^{N} \left[A_n \cosh\frac{n\pi L'}{L} + B_n \sinh\frac{n\pi L'}{L} - \frac{2kL^2[1 + (-1)^{n+1}]}{n^3\pi^3}\right] \sin\frac{n\pi x}{L} - h(x)\right\} \sin\frac{m\pi x}{L}\, dx.$$

These give

$$A_m = \frac{2kL^2[1+(-1)^{m+1}]}{m^3\pi^3} + \frac{2}{L}\int_0^L g(x)\sin\frac{m\pi x}{L}\,dx,$$
$$B_m = \operatorname{csch}\frac{m\pi L'}{L}\left\{\cosh\frac{m\pi L'}{L}\left[-\frac{2kL^2[1+(-1)^{m+1}]}{m^3\pi^3} - \frac{2}{L}\int_0^L g(x)\sin\frac{m\pi x}{L}\,dx\right]\right.$$
$$\left. + \frac{2kL^2[1+(-1)^{m+1}]}{m^3\pi^3} + \frac{2}{L}\int_0^L h(x)\sin\frac{m\pi x}{L}\,dx.\right.$$

This is the N^{th} partial sum of the analytic solution obtained by separation of variables.

The next example cannot be solved with separation of variables; one edge of the region under consideration is not a coordinate curve.

Example 15.4 Use Galerkin's method to find a first approximation to the solution to the following problem involving Poisson's equation on the triangle R in Figure 15.2,

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = F(x,y), \quad (x,y) \text{ in } R, \tag{15.50a}$$
$$V(0,y) = 0, \quad 0 < y < L, \tag{15.50b}$$
$$V(x,0) = 0, \quad 0 < x < L, \tag{15.50c}$$
$$V(x,y) = 0, \quad (x,y) \text{ on } x+y=L. \tag{15.50d}$$

Simplify the approximation when $F(x,y) = k$, a constant.

Solution Since boundary conditions are homogeneous, we take approximations in the form

$$V_N(x,y) = \sum_{n=1}^{N}\sum_{m=1}^{N} c_{nm}\phi_{nm}(x,y),$$

where basis functions $\phi_{nm}(x,y)$ must be linearly independent and from a complete set of functions, and satisfy the boundary conditions. With $\omega(x,y) = xy(L-x-y)$, one possible choice is $\phi_{nm}(x,y) = x^n y^m(L-x-y)$, in which case

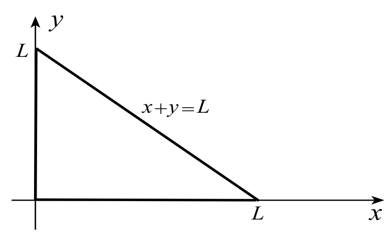

Figure 15.2

$$V_N(x,y) = \sum_{n=1}^{N}\sum_{m=1}^{N} c_{nm}x^n y^m(L-x-y).$$

The first approximation is

$$V_1(x,y) = c_{11}xy(L-x-y),$$

with (equation) residual

$$R = -2c_{11}(x+y) - F(x,y).$$

Galerkin's method requires

$$0 = \int_0^L\int_0^{L-x}[-2c_{11}(x+y) - F(x,y)]xy(L-x-y)\,dy\,dx.$$

Integrations lead to

$$c_{11} = -\frac{90}{L^6}\int_0^L \int_0^{L-x} F(x,y)xy(L-x-y)\,dy\,dx.$$

In the special case that $F(x,y) = k$, a constant, we obtain $c_{11} = -3k/(4L)$, and the first approximation is

$$V_1(x,y) = -\frac{3k}{4L}xy(L-x-y).\bullet$$

In the event that any of the boundary conditions in this example are nonhomogeneous, calculations become more intensive. See Exercise 8 for the case when the nonhomogeneity is along the hypotenuse of the triangle.

EXERCISES 15.5

1. In this exercise we discuss a number of possible ways to approximate the solution to the boundary value problem

$$\begin{aligned} \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} &= 0, \quad 0 < x < L, \quad 0 < y < L', \\ U(0,y) = U(L,y) &= 0, \quad 0 < y < L', \\ U(x,0) &= x(L-x), \quad 0 < x < L, \\ U(x,L') &= 0, \quad 0 < x < L. \end{aligned}$$

(a) Since the function $\phi_0(x,y) = x(L-x)(1-y/L')$ satisfies all four boundary conditions, and the functions $\phi_n(x,y) = x^n y^m (L-x)(L'-y)$, $n,m = 1,2,\ldots$ satisfy homogeneous versions of the boundary conditions, we could take as a first approximation

$$U_1(x,y) = x(L-x)\left(1 - \frac{y}{L'}\right) + cxy(L-x)(L'-y).$$

Use Galerkin's method to determine c.

(b) Since the function $x(L-x)$ satisfies the first three boundary conditions, we could use reduction of dimensionality with $U_1(x,y) = c(y)x(L-x)$. Use Galerkin's method to determine $c(y)$.

(c) The functions $\sin(n\pi x/L)$ satisfy the first two boundary conditions so that we could take approximations in the form

$$U_N(x,y) = \sum_{n=1}^{N} c_n(y)\sin\frac{n\pi x}{L}.$$

Use Galerkin's method to determine the $c_n(y)$. Are approximations partial sums of the analytic solution obtained by separation of variables?

2. The approximation in part (a) of Exercise 1 was available because of the form of the boundary condition along $y = 0$. In addition, this made the calculations in part (b) simpler than they might otherwise be. In this exercise, we replace this boundary condition with $U(x,0) = f(x)$, $0 < x < L$.

(a) Since the function $x(L-x)$ satisfies the first two boundary conditions, we could take a first approximation of the form $U_1(x,y) = c(y)x(L-x)$. Use reduction of dimensionality and Galerkin's method to find $c(y)$. Compare the procedure to that in part (b) of Exercise

1.

(b) Show that approximations of the form in part (c) of Exercise 1 are once again partial sums of the analytic solution obtained by separation of variables.

3. Repeat parts (a) and (b) of Exercise 2 if the boundary condition along $y = L'$ is also nonhomogeneous, $U(x, L') = g(x)$, $0 < x < L$.

4. (a) Could the square in Example 15.2 be divided into two triangles, one above the line $y = x$ and the other below the line, for the subdomain method? Explain.

(b) Divide the square into two triangles one above the line $x + y = L$ and one below for the subdomain method. What does the resulting approximation predict for $V_2(0, 0)$?

5. (a) Pursue approximation 15.48 of problem 15.44 in the case that all boundary conditions are homogeneous ($g(x) = h(x) = 0$).

(b) Confirm that it is the N^{th} partial sum of the analytic solution as obtained by finite Fourier transforms with respect to x and y (see Exercise 54 in Section 7.2).

6. The boundary value problem occurs

$$\frac{\partial^2 V}{\partial x^2} + \epsilon^2 \frac{\partial^2 V}{\partial y^2} = -1, \quad -L < x < L, \quad -L < y < L,$$
$$V(-L, y) = V(L, y) = 0, \quad -L < y < L,$$
$$V(x, -L) = V(x, L) = 0, \quad -L < x < L,$$

where ϵ is a constant, occurs in fluid flow. With $\omega(x, y) = (L^2 - x^2)(L^2 - y^2)$, which vanishes on the boundary of the square, a first polynomial approximation for $V(x, y)$ is $V_1(x, y) = c(L^2 - x^2)(L^2 - y^2)$. Use collocation and Galerkin's method to find c.

7. Use reduction of order to find a first polynomial approximation to problem 15.44 when $F(x, y) = xy$.

8. Find the first approximation to the solution of problem 15.50 when the boundary condition along the hypotenuse of the triangle is $V(x, y) = h(x, y) = h(x, L-x) = g(x)$, where $g(0) = g(L) = 0$.

9. The boundary value problem

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = -2, \quad -L < x < L, \quad -L < y < L,$$
$$V(-L, y) = V(L, y) = 0, \quad -L < y < L,$$
$$V(x, -L) = V(x, L) = 0, \quad -L < x < L,$$

arises in the study of torsion for a square prismatic rod.

(a) With $\omega(x, y) = (L^2 - x^2)(L^2 - y^2)$, which vanishes on the boundary of the square, a first polynomial approximation for $V(x, y)$ is $V_1(x, y) = c(L^2 - x^2)(L^2 - y^2)$. Use Galerkin's method to find c.

(b) The solution $V(x, y)$ must be an even function of x and y, and be symmetric in x and y. Taking this into account, a second polynomial approximation would be $V_2(x, y) = (L^2 - x^2)(L^2 - y^2)[c + d(x^2 + y^2)]$. Use Galerkin's method to find c and d.

(c) Use reduction of dimensionality with $V_1(x) = f(x)(L^2 - y^2)$, where $f(-L) = f(L) = 0$ to approximate the solution.

(d) Use trigonometric basis functions $\cos\dfrac{(2n-1)\pi x}{2L}\cos\dfrac{(2m-1)\pi y}{2L}$, $n, m = 1, 2, \ldots$, to find approximations to $V(x, y)$.

(e) An important integral in this application is

$$M = \int_{-L}^{L}\int_{-L}^{L} V(x, y)\, dy\, dx.$$

To four decimal places, its value is $1.1248L^4$. We can use it to gauge the accuracy of the various approximations. Calculate M for the approximations in parts (a), (b) and (c).

10. We use reduction of dimensionality to approximate solutions of the boundary value problem

$$\begin{aligned}\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} &= 0, \quad 0 < x < L, \quad y > 0,\\ V(0, y) &= 0, \quad y > 0,\\ V(L, y) &= 0, \quad y > 0,\\ V(x, 0) &= x(L - x), \quad 0 < x < L.\end{aligned}$$

Because solutions of this problem must be symmetric about $x = L/2$, choose basis functions $\psi_n(x) = x^n(L - x)^n$ in

$$V_N(x, y) = \sum_{n=1}^{N} c_n(y)\psi_n(x).$$

Such approximations satisfy the homogeneous boundary conditions for arbitrary $c_n(y)$, and satisfy the nonhomogeneous condition provided $c_1(0) = 1$ and $c_n(0) = 0$, $n = 2, \ldots, N$. Find the first approximation $V_1(x, y) = c_1(y)x(L - x)$ using:

(a) collocation;

(b) the subdomain (or moment) method;

(c) Galerkin's method.

11. Use Galerkin's method to find the second approximation in Exercise 10.

12. Since the PDE in Exercise 10 is homogeneous, we might consider a boundary method by choosing basis functions that satisfy the PDE, in particular, $\phi_n(x, y) = e^{-(2n-1)\pi y}\sin\dfrac{(2n-1)\pi x}{L}$. They are also symmetric about $x = L/2$. Approximations are then

$$V_N(x, y) = \sum_{n=1}^{N} c_n e^{-(2n-1)\pi y}\sin\frac{(2n-1)\pi x}{L}.$$

(a) Show that Galerkin's method gives the partial sums of the analytic solution.

(b) Find the first approximation using collocation.

(c) Find the first approximation using the subdomain (or moment) method.

13. Because the nonhomogeneity in Exercise 10 corresponded to the first term in the approximations, it was possible to incorporate the nonhomogeneity into boundary conditions for coefficients $c_n(y)$. This may not always be the case. For instance, consider the same problem where $x(L - x)$ is replaced by an arbitrary function $g(x)$ except that it satisfy $g(0) = g(L) = 0$. With no symmetry about $x = L/2$, basis functions are chosen as $\psi_n(x) = x^n(L - x)$, $n = 1, 2, \ldots$; they satisfy the homogeneous boundary conditions (see Table 15.5). Use Galerkin's method to find:

(a) the first approximation,
(b) the second approximation.

14. Show that when Galerkin's method is used in Exercise 13, with basis functions chosen as eigenfunctions $\psi_n(x) = \sin(n\pi x/L)$ of the associated Sturm-Liouville system, approximations are the partial sums of the analytic solution obtained by separation of variables.

15. Because the PDE in Exercise 13 is homogeneous, we might consider a boundary method by choosing basis functions that satisfy the PDE, in particular, $\phi_n(x,y) = e^{-n\pi y}\sin(n\pi x/L)$. Approximations are then

$$V_N(x,y) = \sum_{n=1}^{N} c_n e^{-n\pi y} \sin\frac{n\pi x}{L}.$$

(a) Show that Galerkin's method gives the partial sums of the analytic solution.
(b) Find the first and second approximations using collocation.
(c) Find the first and second approximations using the subdomain method.
(d) Find the first and second approximations using the moment method.

16. In some developments of the MWR, it is suggested that nonhomogeneous boundary conditions need never be considered; the nonhomogeneity can always be transformed into the PDE. In this exercise we show that whether this is done or not, the same residual to which the MWR would be applied is the same. The residual for problem 15.41 when approximations are taken in form 15.42 where $\phi_0(x,y)$ satisfies the nonhomogeneous boundary condition, and the $\phi_n(x,y)$, $n = 1,\cdots,N$, satisfy the homogeneous version of the boundary condition, is given in equation 15.43. The nonhomogeneity can be removed from the boundary condition with the transformation $V(x,y) = U(x,y) - \phi_0(x,y)$, where again $\phi_0(x,y)$ is a function satisfying the nonhomogeneous boundary condition. With this transformation, problem 15.41 is replaced by

$$L(U) = F(x,y) - L(\phi_i), \quad (x,y) \text{ in } R,$$
$$U(x,y) = 0, \quad (x,y) \text{ on } \beta(R).$$

Show that the residual for approximation $U_N(x,y) = \sum_{n=1}^{N} c_n\phi_n(x,y)$ is also that given in equation 15.43.