## §14.6 Two-dimensional Heat Equation

The homogeneous, two-dimensional heat conduction PDE in some region $R$ of the $xy$-plane is

$$\frac{\partial U}{\partial t} = k\left(\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2}\right). \tag{14.44}$$

Suppose that $R$ is the square $0 \le x \le L$, $0 \le y \le L$, and we discretize this region with a mesh using $N$ equal subdivisions $h = \Delta x = \Delta y = L/N$ in both the $x$ and $y$ directions. With a time step $s = \Delta t$, we discretize the region $0 < x < L$, $0 < y < L$, $t > 0$ in which equation 14.44 is to hold. If $U_{n,m,p} = U(x_n, y_m, t_p)$ denotes approximate valuess for $U(x, y, t)$ at mesh points $(x_n, y_m, t_p)$, the **classic explicit** partial difference equation corresponding to scheme 14.28 is

$$\begin{aligned} U_{n,m,p+1} = U_{n,m,p} + \frac{ks}{h^2}\big(&U_{n+1,m,p} + U_{n,m+1,p} - 4U_{n,m,p} \\ &+ U_{n-1,m,p} + U_{n,m-1,p}\big), \end{aligned} \tag{14.45}$$

$n = 1, \ldots, N-1$, $m = 1, \ldots, N-1$, $p = 0, \ldots$. It uses a forward difference in time and central differences in $x$ and $y$. To determine the stability of this pde, we substitute $E_{n,m,p} = e^{\gamma ps}e^{i\beta(nh+mh)} = e^{\gamma ps}e^{i\beta(n+m)h}$,

$$\begin{aligned} e^{\gamma(p+1)s}e^{i\beta(n+m)h} = e^{\gamma ps}e^{i\beta(n+m)h} + \frac{ks}{h^2}\{&e^{\gamma ps}e^{i\beta(n+1+m)h} + e^{\gamma ps}e^{i\beta(n+m+1)h} \\ &- 4e^{\gamma ps}e^{i\beta(n+m)h} + e^{\gamma ps}e^{i\beta(n-1+m)h} + e^{\gamma ps}e^{i\beta(n+m-1)h}\}. \end{aligned}$$

When we divide by $e^{\gamma ps}e^{i\beta(n+m)h}$, we obtain

$$\begin{aligned} e^{\gamma s} &= 1 + \frac{ks}{h^2}(e^{i\beta h} + e^{i\beta h} - 4 + e^{-i\beta h} + e^{-i\beta h}) = 1 + \frac{4ks}{h^2}(\cos\beta h - 1) \\ &= 1 - \frac{8ks}{h^2}\sin^2\frac{\beta h}{2}. \end{aligned}$$

For Von Neumann stability, condition 14.32 requires

$$\left|1 - \frac{8ks}{h^2}\sin^2\frac{\beta h}{2}\right| \le 1.$$

This reduces to

$$\frac{4ks}{h^2} \le \frac{1}{\sin^2(\beta h/2)}.$$

This will be satisfied for all $\beta$, if

$$\frac{4ks}{h^2} \le 1, \qquad \text{or,} \qquad \Delta t \le \frac{(\Delta x)^2}{4k}. \tag{14.46}$$

In other words, pde 14.45 is conditionally stable.

### Dufort-Frankel Scheme

If we replace the forward time difference with a central difference, and $U_{n,m,p}$ with the average $(U_{n,m,p+1} + U_{n,m,p-1})/2$, we obtain the **Dufort-Frankel** explicit scheme

$$\frac{U_{n,m,p+1}-U_{n,m,p-1}}{2s}=\frac{k}{h^2}\big(U_{n+1,m,p}+U_{n,m+1,p}-2(U_{n,m,p+1}+U_{n,m,p-1})$$
$$+U_{n-1,m,p}+U_{n,m-1,p}\big),$$

or,

$$\left(1+\frac{4ks}{h^2}\right)U_{n,m,p+1}=\left(1-\frac{4ks}{h^2}\right)U_{n,m,p-1}+\frac{2ks}{h^2}(U_{n+1,m,p}+U_{n-1,m,p})$$
$$+\frac{2ks}{h^2}(U_{n,m+1,p}+U_{n,m-1,p}). \qquad (14.47)$$

It is unconditionally stable (Exercise 2). Because it uses values at three time levels, it is necessary to find values at time $t=s$ in order to initiate the scheme. These can be obtained by the classic explicit scheme.

**Backward Implicit Scheme**

If central differences for second derivatives in the classic explicit scheme are centred at $(x_n, y_m, t_{p+1})$ instead of $(x_n, y_m, t_p)$, the result is the **backward implicit** scheme

$$U_{n,m,p+1}=U_{n,m,p}+\frac{ks}{h^2}\left(U_{n+1,m,p+1}+U_{n,m+1,p+1}-4U_{n,m,p+1}+U_{n-1,m,p+1}+U_{n,m-1,p+1}\right),$$

or,

$$\left(1+\frac{4ks}{h^2}\right)U_{n,m,p+1}=U_{n,m,p}+\frac{ks}{h^2}(U_{n+1,m,p+1}+U_{n,m+1,p+1}$$
$$+U_{n-1,m,p+1}+U_{n,m-1,p+1}). \qquad (14.48)$$

Like its one-dimensional counterpart, it is unconditionally stable (Exercise 3).

**Crank-Nicolson Implicit Scheme**

If central differences for second derivatives in the classic explicit scheme are replaced by averages of central differences at time step $t_p$ and at time step $t_{p+1}$, the result is the **Crank-Nicolson implicit** scheme

$$U_{n,m,p+1}=U_{n,m,p}+\frac{ks}{2h^2}\big[(U_{n+1,m,p}+U_{n,m+1,p}-4U_{n,m,p}+U_{n-1,m,p}+U_{n,m-1,p})$$
$$+(U_{n+1,m,p+1}+U_{n,m+1,p+1}-4U_{n,m,p+1}+U_{n-1,m,p+1}+U_{n,m-1,p+1})\big].$$

Rearrangement gives

$$2\left(1+\frac{2ks}{h^2}\right)U_{n,m,p+1}-\frac{ks}{h^2}(U_{n+1,m,p+1}+U_{n,m+1,p+1}+U_{n-1,m,p+1}+U_{n,m-1,p+1})$$
$$=2\left(1-\frac{2ks}{h^2}\right)U_{n,m,p}+\frac{ks}{h^2}(U_{n+1,m,p}+U_{n,m+1,p}+U_{n-1,m,p}+U_{n,m-1,p}). \qquad (14.49)$$

It is also unconditionally stable (Exercise 4).

### Irregular Shaped Regions

Finite differences, and even more so finite elements, show their indispensability when PDEs are to be considered on regions whose boundaries are not coordinate curves (in the plane) and coordinate surfaces (in space). For example, suppose PDE 14.44 is to describe heat flow in the elliptical plate of Figure 14.10. None of our analytic techniques are applicable to this problem. Finite differences can be adapted to irregular boundaries, with some difficulties, but the difficulties have more to do with computer implementation than with the theoretical aspects of the adaptation.

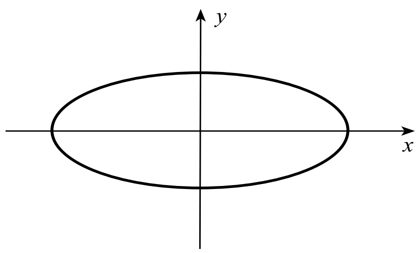


**Figure 14.10**

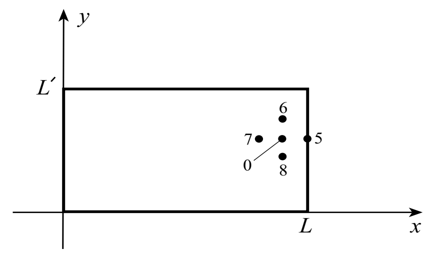


**Figure 14.11**

When the region is a rectangle (Figure 14.11), a central difference for the Laplacian at node 0 closest to boundary $x = L$ utilizes the boundary data at node 5. When the boundary is curved (Figure 14.12), a discretization of the region with the usual array of points results in very few mesh points on the boundary of the region. A central difference at 0 has node 5 (and node 6) outside the region. We need to replace the "central" difference formula for the Laplacian at node 0 with a difference formula that utilizes boundary data at nodes 1 and 2 in place of nodes 5 and 6. More generally, we need a difference formula that accommodates two horizontal and two vertical nodes at differing distances from node 0. We have shown this in Figure 14.13 where all four surrounding nodes are at different distances from node 0.

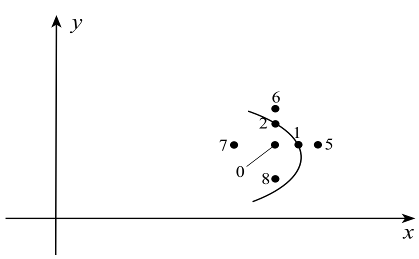


**Figure 14.12**

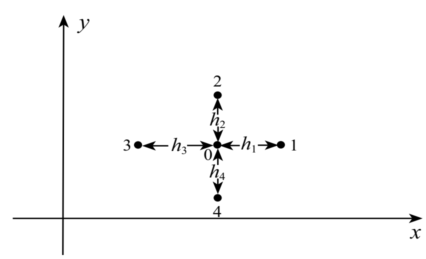


**Figure 14.13**

Suppose we denote the values of the function $U(x, y)$ at the five nodes by $U(0), \ldots,$ $U(4)$. We seek an approximation to the Laplacian of $U(x, y)$ at node 0 as a linear combination of $U(0), \ldots, U(4)$,

$$(U_{xx} + U_{yy})_{|\text{node } 0} = \sum_{i=0}^{4} \alpha_i U(i). \tag{14.50}$$

To find suitable constants $\alpha_i$, we represent $U(x, y)$ at nodes 1, 2, 3, and 4 in Taylor series at node 0. If we extend the notation $U(i)$, to include derivatives, such as $U_x(0)$, then

$$\begin{aligned}
U(1) &= U(0) + U_x(0)h_1 + \frac{1}{2}U_{xx}(0)h_1^2 + \frac{1}{3!}U_{xxx}(0)h_1^3 + \cdots,\\
U(2) &= U(0) + U_y(0)h_2 + \frac{1}{2}U_{yy}(0)h_2^2 + \frac{1}{3!}U_{yyy}(0)h_2^3 + \cdots,\\
U(3) &= U(0) - U_x(0)h_3 + \frac{1}{2}U_{xx}(0)h_3^2 - \frac{1}{3!}U_{xxx}(0)h_3^3 + \cdots,\\
U(4) &= U(0) - U_y(0)h_4 + \frac{1}{2}U_{yy}(0)h_4^2 - \frac{1}{3!}U_{yyy}(0)h_4^3 + \cdots.
\end{aligned}$$

When we substitute these into equation 14.50, and gather like terms, the result is

$$\begin{aligned}
(U_{xx} + U_{yy})_{\text{node }0} &= (\alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4)U(0) + (\alpha_1 h_1 - \alpha_3 h_3)U_x(0) + (\alpha_2 h_2 - \alpha_4 h_4)U_y(0)\\
&\quad + \frac{1}{2}(\alpha_1 h_1^2 + \alpha_3 h_3^2)U_{xx}(0) + \frac{1}{2}(\alpha_2 h_2^2 + \alpha_4 h_4^2)U_{yy}(0) + \cdots .
\end{aligned}$$

For the the right side to agree with the left, we require

$$\begin{aligned}
\alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 &= 0,\\
\alpha_1 h_1 - \alpha_3 h_3 &= 0,\\
\alpha_2 h_2 - \alpha_4 h_4 &= 0,\\
\alpha_1 h_1^2 + \alpha_3 h_3^2 &= 2,\\
\alpha_2 h_2^2 + \alpha_4 h_4^2 &= 2.
\end{aligned}$$

The solution of these equations is

$$\alpha_0 = -2\left(\frac{1}{h_1 h_3} + \frac{1}{h_2 h_4}\right), \qquad \alpha_1 = \frac{2}{h_1(h_1 + h_3)}, \qquad \alpha_2 = \frac{2}{h_2(h_2 + h_4)},$$

$$\alpha_3 = \frac{2}{h_3(h_1 + h_3)}, \qquad \alpha_4 = \frac{2}{h_4(h_2 + h_4)}.$$

Thus, a difference formula for the Laplacian of $U(x, y)$ at node 0 in Figure 14.13 in terms of values of the function at the five nodes is

$$\begin{aligned}
(U_{xx} + U_{yy})_{|\text{node }0} &= -2\left(\frac{1}{h_1 h_3} + \frac{1}{h_2 h_4}\right)U(0) + \frac{2U(1)}{h_1(h_1 + h_3)} + \frac{2U(2)}{h_2(h_2 + h_4)}\\
&\quad + \frac{2U(3)}{h_3(h_1 + h_3)} + \frac{2U(4)}{h_4(h_2 + h_4)}.
\end{aligned} \tag{14.51}$$

The reader can perhaps appreciate that the computer implementation of this formula at each node of the region of Figure 14.44, wherein it is required, could be a programming nightmare.

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## EXERCISES 14.6

**1.** Generalize pde 14.45 to a rectangle $0 \le x \le L$, $0 \le y \le L'$. What is the stability condition replacing inequality 14.46?

**2.** Verify that the Dufort-Frankel scheme 14.47 is stable.

**3.** Verify that the backward implicit scheme 14.48 is stable.

**4.** Verify that the Crank-Nicolson scheme 14.49 is stable.