MATH 1010 Assignment 3 Winter 2014

Solutions to Assignment 3

1. (a) This matrix is in row echelon form.

(b) This matrix is not in row echelon form.

$$\left(\begin{array}{ccc|c} 2 & 1 & 3 & 4 \\ 0 & 1 & 4 & 3 \\ 0 & 2 & 1 & 4 \end{array}\right) R_3 \to -2R_2 + R_3 \quad \to \left(\begin{array}{ccc|c} 2 & 1 & 3 & 4 \\ 0 & 1 & 4 & 3 \\ 0 & 0 & -7 & -2 \end{array}\right) \begin{array}{c} R_1 \to R_1/2 \\ \\ R_3 \to R_3/(-7) \end{array} \to \left(\begin{array}{ccc|c} 1 & 1/2 & 3/2 & 2 \\ 0 & 1 & 4 & 3 \\ 0 & 0 & 1 & 2/7 \end{array}\right)$$

(c) This matrix is not in row echelon form.

$$\left(\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 4 & 3 \\ 0 & 0 & 0 \\ 0 & 8 & 6 \end{array}\right) \begin{array}{c} R_2 \to R_2/4 \\ \\ R_4 \to -2R_2 + R_4 \end{array} \to \left(\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 3/4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

2. (a) This matrix is not in reduced row echelon form.

$$\left(\begin{array}{ccc|c} 1 & 2 & -3 & 4 \\ 0 & 0 & 5 & 1 \\ 0 & 1 & 2 & 2 \end{array}\right) \begin{array}{c} R_2 \to R_3 \\ R_3 \to R_2 \end{array} \to \left(\begin{array}{ccc|c} 1 & 2 & -3 & 4 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 5 & 1 \end{array}\right) \begin{array}{c} R_1 \to -2R_2 + R_1 \\ \\ R_3 \to R_3/5 \end{array}$$

$$\to \left(\begin{array}{ccc|c} 1 & 0 & -7 & 0 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 1 & 1/5 \end{array}\right) \begin{array}{c} R_1 \to 7R_3 + R_1 \\ R_2 \to -2R_3 + R_2 \end{array} \to \left(\begin{array}{ccc|c} 1 & 0 & 0 & 7/5 \\ 0 & 1 & 0 & 8/5 \\ 0 & 0 & 1 & 1/5 \end{array}\right)$$

(b) This matrix is in reduced row echelon form.

(c) This matrix is not in reduced row echelon form.

$$\left(\begin{array}{ccc|c} 3 & 0 & 1 & -6 \\ 0 & 1 & 2 & 5 \\ 0 & 0 & 4 & 4 \end{array}\right) R_3 \to R_3/4 \quad \to \left(\begin{array}{ccc|c} 3 & 0 & 1 & -6 \\ 0 & 1 & 2 & 5 \\ 0 & 0 & 1 & 1 \end{array}\right) \begin{array}{c} R_1 \to -R_3 + R_1 \\ R_2 \to -2R_3 + R_2 \end{array}$$

$$\to \left(\begin{array}{ccc|c} 3 & 0 & 0 & -7 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 1 \end{array}\right) R_1 \to R_1/3 \quad \to \left(\begin{array}{ccc|c} 1 & 0 & 0 & -7/3 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 1 \end{array}\right)$$

3. (a) $\left(\begin{array}{ccc|c} 2 & 1 & -1 & 6 \\ 3 & -2 & 4 & 7 \\ 7 & 0 & 2 & 1 \end{array}\right) R_2 \to -R_1 + R_2 \to \left(\begin{array}{ccc|c} 2 & 1 & -1 & 6 \\ 1 & -3 & 5 & 1 \\ 7 & 0 & 2 & 1 \end{array}\right) \begin{array}{c} R_1 \to R_2 \\ R_2 \to R_1 \end{array}$

$$\to \left(\begin{array}{ccc|c} 1 & -3 & 5 & 1 \\ 2 & 1 & -1 & 6 \\ 7 & 0 & 2 & 1 \end{array}\right) \begin{array}{c} R_2 \to -2R_1 + R_2 \\ R_3 \to -7R_1 + R_3 \end{array} \to \left(\begin{array}{ccc|c} 1 & -3 & 5 & 1 \\ 0 & 7 & -11 & 4 \\ 0 & 21 & -33 & -6 \end{array}\right) R_3 \to -3R_2 + R_3$$

$$\to \left(\begin{array}{ccc|c} 1 & -3 & 5 & 1 \\ 0 & 7 & -11 & 4 \\ 0 & 0 & 0 & -18 \end{array}\right)$$

When we convert to equations, the last line gives $0 = -18$. Hence, the equations do not have a solution.

(b) $$\left(\begin{array}{ccc|c} 3 & 2 & 2 & 4 \\ 2 & -1 & 4 & 8 \\ 4 & 2 & -1 & 6 \end{array}\right) \begin{array}{l} R_1 \to -R_2 + R_1 \\ \\ \\ \end{array} \to \left(\begin{array}{ccc|c} 1 & 3 & -2 & -4 \\ 2 & -1 & 4 & 8 \\ 4 & 2 & -1 & 6 \end{array}\right) \begin{array}{l} \\ R_2 \to -2R_1 + R_2 \\ R_3 \to -4R_1 + R_3 \end{array}$$

$$\to \left(\begin{array}{ccc|c} 1 & 3 & -2 & -4 \\ 0 & -7 & 8 & 16 \\ 0 & -10 & 7 & 22 \end{array}\right) \begin{array}{l} \\ \\ R_3 \to -R_2 + R_3 \end{array} \to \left(\begin{array}{ccc|c} 1 & 3 & -2 & -4 \\ 0 & -7 & 8 & 16 \\ 0 & -3 & -1 & 6 \end{array}\right) R_2 \to -2R_3 + R_2$$

$$\to \left(\begin{array}{ccc|c} 1 & 3 & -2 & -4 \\ 0 & -1 & 10 & 4 \\ 0 & -3 & -1 & 6 \end{array}\right) \begin{array}{l} \\ R_2 \to -R_2 \\ R_3 \to -R_3 \end{array} \to \left(\begin{array}{ccc|c} 1 & 3 & -2 & -4 \\ 0 & 1 & -10 & -4 \\ 0 & 3 & 1 & -6 \end{array}\right) \begin{array}{l} \\ \\ R_3 \to -3R_2 + R_3 \end{array}$$

$$\to \left(\begin{array}{ccc|c} 1 & 3 & -2 & -4 \\ 0 & 1 & -10 & -4 \\ 0 & 0 & 31 & 6 \end{array}\right) \begin{array}{l} \\ \\ R_3 \to R_3/31 \end{array} \to \left(\begin{array}{ccc|c} 1 & 3 & -2 & -4 \\ 0 & 1 & -10 & -4 \\ 0 & 0 & 1 & 6/31 \end{array}\right)$$

When we convert to equations, we obtain

$$\begin{aligned} x + 3y - 2z &= -4, \\ y - 10z &= -4, \\ z &= 6/31. \end{aligned}$$

Back substitutions now give

$$\begin{aligned} y &= -4 + 10z = -4 + \frac{60}{31} = -\frac{64}{31}, \\ x &= -4 - 3y + 2z = -4 - 3\left(-\frac{64}{31}\right) + 2\left(\frac{6}{31}\right) = \frac{80}{31}. \end{aligned}$$

The solution is $x = 80/31$, $y = -64/31$, $z = 6/31$.

4. (a) $$\left(\begin{array}{ccc|c} 1 & 2 & -1 & 4 \\ 3 & -2 & 2 & -1 \\ 10 & -4 & 2 & 5 \end{array}\right) \begin{array}{l} \\ R_2 \to -3R_1 + R_2 \\ R_3 \to -10R_1 + R_3 \end{array} \to \left(\begin{array}{ccc|c} 1 & 2 & -1 & 4 \\ 0 & -8 & 5 & -13 \\ 0 & -24 & 12 & -35 \end{array}\right) \begin{array}{l} \\ \\ R_3 \to -3R_2 + R_3 \end{array}$$

$$\to \left(\begin{array}{ccc|c} 1 & 2 & -1 & 4 \\ 0 & -8 & 5 & -13 \\ 0 & 0 & -3 & 4 \end{array}\right) \begin{array}{l} \\ \\ R_3 \to R_3/(-3) \end{array} \to \left(\begin{array}{ccc|c} 1 & 2 & -1 & 4 \\ 0 & -8 & 5 & -13 \\ 0 & 0 & 1 & -4/3 \end{array}\right) \begin{array}{l} R_1 \to R_3 + R_1 \\ R_2 \to -5R_3 + R_2 \\ \\ \end{array}$$

$$\to \left(\begin{array}{ccc|c} 1 & 2 & 0 & 8/3 \\ 0 & -8 & 0 & -19/3 \\ 0 & 0 & 1 & -4/3 \end{array}\right) R_2 \to R_2/(-8) \to \left(\begin{array}{ccc|c} 1 & 2 & 0 & 8/3 \\ 0 & 1 & 0 & 19/24 \\ 0 & 0 & 1 & -4/3 \end{array}\right) \begin{array}{l} R_1 \to -2R_2 + R_1 \\ \\ \\ \end{array}$$

$$\to \left(\begin{array}{ccc|c} 1 & 0 & 0 & 13/12 \\ 0 & 1 & 0 & 19/24 \\ 0 & 0 & 1 & -4/3 \end{array}\right)$$

When we convert to equations, we obtain the solution $x = 13/12$, $y = 19/24$, and $z = -4/3$.

(b) $\left(\begin{array}{cccc|c} 1 & 1 & -2 & 2 & 1 \\ 3 & 0 & -1 & 5 & 2 \\ 4 & 1 & -2 & 2 & 2 \\ -2 & -2 & 3 & 1 & -1 \end{array}\right) \begin{array}{l} R_2 \to -3R_1 + R_2 \\ R_3 \to -4R_1 + R_3 \\ R_4 \to 2R_1 + R_4 \end{array} \to \left(\begin{array}{cccc|c} 1 & 1 & -2 & 2 & 1 \\ 0 & -3 & 5 & -1 & -1 \\ 0 & -3 & 6 & -6 & -2 \\ 0 & 0 & -1 & 5 & 1 \end{array}\right) R_3 \to -R_2 + R_3$

$\to \left(\begin{array}{cccc|c} 1 & 1 & -2 & 2 & 1 \\ 0 & -3 & 5 & -1 & -1 \\ 0 & 0 & 1 & -5 & -1 \\ 0 & 0 & -1 & 5 & 1 \end{array}\right) \begin{array}{l} R_1 \to 2R_3 + R_1 \\ R_2 \to -5R_3 + R_2 \\ \\ R_4 \to R_3 + R_4 \end{array} \to \left(\begin{array}{cccc|c} 1 & 1 & 0 & -8 & -1 \\ 0 & -3 & 0 & 24 & 4 \\ 0 & 0 & 1 & -5 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) R_2 \to R_2/(-3)$

$\to \left(\begin{array}{cccc|c} 1 & 1 & 0 & -8 & -1 \\ 0 & 1 & 0 & -8 & -4/3 \\ 0 & 0 & 1 & -5 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) R_1 \to -R_2 + R_1 \to \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1/3 \\ 0 & 1 & 0 & -8 & -4/3 \\ 0 & 0 & 1 & -5 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$

When we convert to equations,

$$x = \frac{1}{3}, \quad y - 8w = -\frac{4}{3}, \quad z - 5w = -1.$$

The solution is $x = \frac{1}{3}$, $y = -\frac{4}{3} + 8w$, $z = -1 + 5w$, where w is arbitrary.

5. (a) $\left(\begin{array}{ccc|ccc} 2 & 1 & 3 & 1 & 0 & 0 \\ 1 & -1 & 2 & 0 & 1 & 0 \\ -1 & 5 & 0 & 0 & 0 & 1 \end{array}\right) \begin{array}{l} R_1 \to R_2 \\ R_2 \to R_1 \end{array} \to \left(\begin{array}{ccc|ccc} 1 & -1 & 2 & 0 & 1 & 0 \\ 2 & 1 & 3 & 1 & 0 & 0 \\ -1 & 5 & 0 & 0 & 0 & 1 \end{array}\right) \begin{array}{l} R_2 \to -2R_1 + R_2 \\ R_3 \to R_1 + R_3 \end{array}$

$\to \left(\begin{array}{ccc|ccc} 1 & -1 & 2 & 0 & 1 & 0 \\ 0 & 3 & -1 & 1 & -2 & 0 \\ 0 & 4 & 2 & 0 & 1 & 1 \end{array}\right) R_3 \to -R_2 + R_3 \to \left(\begin{array}{ccc|ccc} 1 & -1 & 2 & 0 & 1 & 0 \\ 0 & 3 & -1 & 1 & -2 & 0 \\ 0 & 1 & 3 & -1 & 3 & 1 \end{array}\right) \begin{array}{l} R_2 \to R_3 \\ R_3 \to R_2 \end{array}$

$\to \left(\begin{array}{ccc|ccc} 1 & -1 & 2 & 0 & 1 & 0 \\ 0 & 1 & 3 & -1 & 3 & 1 \\ 0 & 3 & -1 & 1 & -2 & 0 \end{array}\right) \begin{array}{l} R_1 \to R_2 + R_1 \\ \\ R_3 \to -3R_2 + R_3 \end{array} \to \left(\begin{array}{ccc|ccc} 1 & 0 & 5 & -1 & 4 & 1 \\ 0 & 1 & 3 & -1 & 3 & 1 \\ 0 & 0 & -10 & 4 & -11 & -3 \end{array}\right) R_3 \to R_3/(-10)$

$\to \left(\begin{array}{ccc|ccc} 1 & 0 & 5 & -1 & 4 & 1 \\ 0 & 1 & 3 & -1 & 3 & 1 \\ 0 & 0 & 1 & -2/5 & 11/10 & 3/10 \end{array}\right) \begin{array}{l} R_1 \to -5R_3 + R_1 \\ R_2 \to -3R_3 + R_2 \end{array} \to \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -3/2 & -1/2 \\ 0 & 1 & 0 & 1/5 & -3/10 & 1/10 \\ 0 & 0 & 1 & -2/5 & 11/10 & 3/10 \end{array}\right)$

The inverse matrix is $A^{-1} = \begin{pmatrix} 1 & -3/2 & -1/2 \\ 1/5 & -3/10 & 1/10 \\ -2/5 & 11/10 & 3/10 \end{pmatrix} = \frac{1}{10}\begin{pmatrix} 10 & -15 & -5 \\ 2 & -3 & 1 \\ -4 & 11 & 3 \end{pmatrix}$.

(b) $\left(\begin{array}{ccc|ccc} 3 & 2 & -1 & 1 & 0 & 0 \\ 4 & 5 & 2 & 0 & 1 & 0 \\ -2 & 1 & 4 & 0 & 0 & 1 \end{array}\right) R_1 \to R_3 + R_1 \to \left(\begin{array}{ccc|ccc} 1 & 3 & 3 & 1 & 0 & 1 \\ 4 & 5 & 2 & 0 & 1 & 0 \\ -2 & 1 & 4 & 0 & 0 & 1 \end{array}\right) \begin{array}{l} R_2 \to -4R_1 + R_2 \\ R_3 \to 2R_1 + R_3 \end{array}$

$\to \left(\begin{array}{ccc|ccc} 1 & 3 & 3 & 1 & 0 & 1 \\ 0 & -7 & -10 & -4 & 1 & -4 \\ 0 & 7 & 10 & 2 & 0 & 3 \end{array}\right) R_3 \to R_2 + R_3 \to \left(\begin{array}{ccc|ccc} 1 & 3 & 3 & 1 & 0 & 1 \\ 0 & -7 & -10 & -4 & 1 & -4 \\ 0 & 0 & 0 & -2 & 1 & -1 \end{array}\right)$

This shows that the matrix does not have an inverse.