MATH 1210 Solutions to Assignment 1 Fall 2018

1. (a) Use mathematical induction to prove that

$$1^2+4^2+7^2+10^2+13^2+\cdots+(6n+1)^2=(2n+1)(12n^2+9n+1), \qquad n\geq 1.$$

(b) You were able to prove that the sum of the terms on the left side of the equation in part (a) is given by the formula on the right because we gave the formula to you. Suppose now that the formula is not given to you, but you have to find it. Do so using sigma notation.

(a) When $n=1$ the left side is $1^2+4^2+7^2=66$, and the formula gives $(3)(12+9+1)=66$. Thus, the formula is correct for $n=1$. Suppose that the formula is valid for some integer k; that is,

$$1^2+4^2+7^2+10^2+13^2+\cdots+(6k+1)^2=(2k+1)(12k^2+9k+1).$$

We must now prove that the formula is correct for $k+1$; that is, we must prove that

$$\begin{aligned}1^2+4^2+7^2+10^2+13^2+\cdots+[6(k+1)+1]^2&=[2(k+1)+1][12(k+1)^2+9(k+1)+1]\\&=(2k+3)(12k^2+33k+22)=24k^3+102k^2+143k+66.\end{aligned}$$

The summation on the left is

$$\begin{aligned}1^2+4^2+7^2+10^2+13^2+\cdots+(6k+7)^2&=[1^2+4^2+7^2+10^2+13^2+\cdots+(6k+1)^2]+(6k+4)^2+(6k+7)^2\\&=(2k+1)(12k^2+9k+1)+(36k^2+48k+16)+(36k^2+84k+49)\\&=(24k^3+30k^2+11k+1)+(72k^2+132k+65)\\&=24k^3+102k^2+143k+66.\end{aligned}$$

Thus, the formula is correct for $k+1$, and by the principle of mathematical induction, it is valid for all $n\geq 1$.

(b) When we write the summation in sigma notation, we get

$$\begin{aligned}\sum_{i=1}^{2n+1}(3i-2)^2=\sum_{i=1}^{2n+1}(9i^2-12i+4)&=9\sum_{i=1}^{2n+1}i^2-12\sum_{i=1}^{2n+1}i+4\sum_{i=1}^{2n+1}1\\&=9\left[\frac{(2n+1)(2n+2)(4n+3)}{6}\right]-12\left[\frac{(2n+1)(2n+2)}{2}\right]+4(2n+1)\\&=3(2n+1)(n+1)(4n+3)-6(2n+1)(2n+2)+4(2n+1)\\&=(2n+1)[3(n+1)(4n+3)-6(2n+2)+4]\\&=(2n+1)(12n^2+21n+9-12n-12+4)\\&=(2n+1)(12n^2+9n+1).\end{aligned}$$

2. Prove that $3^n + 7^n - 2$ is divisible by 8 for all $n \geq 1$.

When $n = 1$, $3^n + 7^n - 2 = 8$, which is divisible by 8. The result is therefore valid for $n = 1$. Suppose that $3^k + 7^k - 2$ is divisible by 8. We must now show that $3^{k+1} + 7^{k+1} - 2$ is divisible by 8. We write that

$$\begin{aligned} 3^{k+1} + 7^{k+1} - 2 = 3 \cdot 3^k + 7 \cdot 7^k - 2 &= 3(3^k + 7^k - 2) - 3 \cdot 7^k + 6 + 7 \cdot 7^k - 2 \\ &= 3(3^k + 7^k - 2) + 4 \cdot 7^k + 4 = 3(3^k + 7^k - 2) + 4(7^k + 1). \end{aligned}$$

Now $3^k + 7^k - 2$ is divisible by 8. Furthermore, $7^n + 1$ is always an even integer, and is therefore divisible by 2. Hence, $4(7^k + 1)$ is divisible by 8. It follows that the right side of the equation is divisible by 8, and therefore so also must be $3^{k+1} + 7^{k+1} - 2$. The result is valid for $k + 1$, and by the principle of mathematical induction, it is valid for all $n \geq 1$.

3. (a) Use mathematical induction to prove that $n^2 > 2n + 1$ for all $n \geq 3$.
(b) Use the result in part (a) to prove that $2^n > n^2$ for $n \geq 5$.

(a) Certainly $3^2 > 2(3) + 1$, so that the result is true for $n = 3$. Suppose that $k^2 > 2k + 1$ for some integer k. We must now show that $(k+1)^2 > 2(k+1) + 1 = 2k + 3$. Now,

$$(k+1)^2 = k^2 + 2k + 1 > (2k+1) + 2k + 1 = 4k + 2 = (2k+3) + (2k-1) > 2k + 3.$$

Thus, the result is valid for $k + 1$, and is therefore correct for all $n \geq 3$ (by the principle of mathematical induction).
(b) Certainly, $2^5 > 5^2$, so that the inequality is valid for $n = 5$. Suppose that $2^k > k^2$ for some integer k. We must show that $2^{k+1} > (k+1)^2$. Now,

$$\begin{aligned} 2^{k+1} &= 2 \cdot 2^k > 2(k^2) = 2k^2 = k^2 + k^2 \\ &> k^2 + 2k + 1 \qquad\qquad \text{(by part (a))} \\ &= (k+1)^2. \end{aligned}$$

Consequently, the inequality is correct for $k + 1$, and by the principle of mathematical induction, it is valid for all $n \geq 5$.

4. Find a formula for

$$\sum_{i=1}^{n} \frac{1}{4i^2-1}.$$

Hint: See Exercise 34 in Section 1.2.

We write that

$$\frac{1}{4i^2-1} = \frac{1}{(2i+1)(2i-1)} = \frac{1/2}{2i-1} - \frac{1/2}{2i+1} = \frac{1}{2}\left(\frac{1}{2i-1} - \frac{1}{2i+1}\right).$$

With this, we can write terms in the sum in the following form

$$\begin{aligned}\sum_{i=1}^{n} \frac{1}{4i^2-1} &= \frac{1}{2}\sum_{i=1}^{n}\left(\frac{1}{2i-1} - \frac{1}{2i+1}\right)\\ &= \frac{1}{2}\left[\left(1-\frac{1}{3}\right) + \left(\frac{1}{3}-\frac{1}{5}\right) + \left(\frac{1}{5}-\frac{1}{7}\right) + \cdots + \left(\frac{1}{2n-1} - \frac{1}{2n+1}\right)\right].\end{aligned}$$

Most terms cancel and what remains is

$$\sum_{i=1}^{n} \frac{1}{4i^2-1} = \frac{1}{2}\left(1 - \frac{1}{2n+1}\right) = \frac{n}{2n+1}.$$

5. Evaluate

$$\sum_{j=m}^{n} (2j-1)^2$$

for $n > m$. It is not necessary for you to simplify your result.

$$\begin{aligned}\sum_{j=m}^{n} (2j-1)^2 &= \sum_{j=1}^{n} (2j-1)^2 - \sum_{j=1}^{m-1} (2j-1)^2\\ &= \sum_{j=1}^{n} (4j^2-4j+1) - \sum_{j=1}^{m-1} (4j^2-4j+1)\\ &= 4\sum_{j=1}^{n} j^2 - 4\sum_{j=1}^{n} j + \sum_{j=1}^{n} 1 - 4\sum_{j=1}^{m-1} j^2 + 4\sum_{j=1}^{m-1} j - \sum_{j=1}^{m-1} 1\\ &= 4\left[\frac{n(n+1)(2n+1)}{6}\right] - 4\left[\frac{n(n+1)}{2}\right] + n\\ &\qquad - 4\left[\frac{(m-1)(m)(2m-1)}{6}\right] + 4\left[\frac{(m-1)(m)}{2}\right] - (m-1).\end{aligned}$$

6. Simplify each of the following complex expressions to Cartesian form:

$$\text{(a)} \quad \frac{i^{15}(3+2i)^2}{1-i} \qquad\qquad \text{(b)} \quad \frac{i^6(1+2\bar{i})\overline{(3-4i)}}{3+\overline{2i}}$$

$$\text{(a)} \ \frac{i^{15}(3+2i)^2}{1-i} = \frac{(-i)(5+12i)}{1-i} = \frac{(12-5i)(1+i)}{(1-i)(1+i)} = \frac{17+7i}{2} = \frac{17}{2} + \frac{7}{2}i$$

$$\text{(b)} \ \frac{i^6(1+2\bar{i})\overline{(3-4i)}}{3+\overline{2i}} = \frac{(-1)(1-2i)(3+4i)}{3-2i} = \frac{(-11+2i)(3+2i)}{(3-2i)(3+2i)} = \frac{-37-16i}{13} = -\frac{37}{13} - \frac{16}{13}i$$

7. Find all solutions for each of the following equations:

$$\text{(a)} \quad 2z^2+3z+15=0 \qquad \text{(b)} \quad z^4+3z^2-5=0 \qquad \text{(c)} \quad z^5 = 32i$$

(a) Using the quadratic formula,

$$z = \frac{-3 \pm \sqrt{9-120}}{4} = \frac{-3 \pm \sqrt{-111}}{4} = \frac{-3 \pm \sqrt{111}i}{4}.$$

(b) If we set $w = z^2$, then

$$w^2 + 3w - 5 = 0.$$

Solutions of this equation are

$$w = \frac{-3 \pm \sqrt{9+20}}{2} = \frac{-3 \pm \sqrt{29}}{2}.$$

Thus,

$$z^2 = \frac{-3+\sqrt{29}}{2} \qquad \text{or} \qquad z^2 = \frac{-3-\sqrt{29}}{2}.$$

When we take square roots,

$$z = \pm\sqrt{\frac{-3+\sqrt{29}}{2}} \qquad \text{or} \qquad z = \pm\sqrt{\frac{3+\sqrt{29}}{2}}i.$$

(c) We begin by writing $32i$ in exponential form

$$z^5 = 32i = 32e^{\pi i/2} = 32e^{\pi i/2}e^{2k\pi i} = 32e^{(1/2+2k)\pi i}.$$

We now take fifth roots of both sides of the equation

$$z = [32e^{(1/2+2k)\pi i}]^{1/5} = 2e^{(1/10+2k/5)\pi i}.$$

When we set $k = 0, 1, 2, 3, 4$, we get the five solutions of the equation:

$$z_0 = 2e^{\pi i/10} = 2\cos\left(\frac{\pi}{10}\right) + 2\sin\left(\frac{\pi}{10}\right)i,$$
$$z_1 = 2e^{\pi i/2} = 2i,$$

$$z_2 = 2e^{9\pi i/10} = 2\cos\left(\frac{9\pi}{10}\right) + 2\sin\left(\frac{9\pi}{10}\right)i,$$
$$z_3 = 2e^{13\pi i/10} = 2\cos\left(\frac{13\pi}{10}\right) + 2\sin\left(\frac{13\pi}{10}\right)i,$$
$$z_4 = 2e^{17\pi i/10} = 2\cos\left(\frac{17\pi}{10}\right) + 2\sin\left(\frac{17\pi}{10}\right)i.$$