Solutions to Assignment 1

1. (a) When $n = 2$:

L.S.$=5 + 6 + 7 = 18$ R.S.$=2(3)(3) = 18$

Thus, the result is valid for $n = 2$.

(b)When $n = 3$:

L.S.$=6 + 7 + 8 + 9 + 10 = 40$ R.S.$=2(4)(5) = 40$

Thus, the result is also valid for $n = 3$.

(c) When $n = 1$:

L.S.$=4$ R.S$=2(2)(1) = 4$.

Thus, the result is valid for $n = 1$. Assume the result is valid for some integer k; that is,

$$(k + 3) + (k + 4) + (k + 5) + \cdots + (3k + 1) = 2(k + 1)(2k - 1).$$

We now want to prove that the result is valid for $k + 1$; that is,

$$(k + 4) + (k + 5) + (k + 6) + \cdots + (3k + 4) = 2(k + 2)(2k + 1).$$

The left side can be rewritten in the form

$$\begin{aligned}
\text{L.S.} &= [(k + 3) + (k + 4) + (k + 5) + \cdots + (3k + 1)] \\
&\qquad + (3k + 2) + (3k + 3) + (3k + 4) - (k + 3) \\
&= 2(k + 1)(2k - 1) + (3k + 2) + (3k + 3) + (3k + 4) - (k + 3) \\
&= 2(k + 1)(2k - 1) + (8k + 6) \\
&= 4k^2 + 10k + 4 \\
&= 2(k + 2)(2k + 1) = \text{R.S.}.
\end{aligned}$$

This proves the result for $k+1$, and therefore, by mathematical induction, the result is valid for all $n \geq 1$.

2. (a) When $n = 3$, $4^n + 6n - 1$ becomes $4^3 + 6(3) - 1 = 81$, which is divisible by 9. When $n = 4$, $4^n + 6n - 1$ becomes $4^4 + 6(4) - 1 = 279$, which is divisible by 9.

(b) When $n = 1$, $4^n + 6(n) - 1$ becomes $4^1 + 6(1) - 1 = 9$, which is divisible by 9. Assume for some integer k that $4^k + 6k - 1$ is divisible by 9. Consider

$$\begin{aligned} 4^{k+1} + 6(k+1) - 1 &= 4(4^k) + 6k + 5 = 4(4^k + 6k - 1) + 6k + 5 - 24k + 4 \\ &= 4(4^k + 6k - 1) - 18k + 9 = 4(4^k + 6k - 1) - 9(2k - 1). \end{aligned}$$

Since both terms on the right are divisible by 9, it follows that $4^{k+1} + 6(k+1) - 1$ is divisible by 9; that is, the result is valid for $k + 1$. Hence, by mathematical induction, $4^n + 6n - 1$ is divisible by 9 for all $n \geq 1$.

3. Let us lower the limits of summation by 6, compensating by raising m's by 6 after the sigma sign,

$$\begin{aligned} S &= \sum_{m=1}^{17} (m + 6 - 1)[(m+6)^2 + 5] = \sum_{m=1}^{17} (m+5)(m^2 + 12m + 41) \\ &= \sum_{m=1}^{17} (m^3 + 17m^2 + 101m + 205) \\ &= \sum_{m=1}^{17} m^3 + 17 \sum_{m=1}^{17} m^2 + 101 \sum_{m=1}^{17} m + 205 \sum_{n=1}^{17} 1 \\ &= \left(\frac{17 \cdot 18}{2}\right)^2 + 17\left(\frac{17 \cdot 18 \cdot 35}{6}\right) + 101\left(\frac{17 \cdot 18}{2}\right) + 205(17) \\ &= 72\,692. \end{aligned}$$

The sum of the digits is $7 + 2 + 6 + 9 + 2 = 26$.

4. (a)

$$\begin{aligned} w &= \frac{(1+i)^3}{3+2i} + \frac{1}{1-i} = \frac{1 + 3i - 3 - i}{3+2i} + \frac{1}{1-i} = \frac{-2+2i}{3+2i} + \frac{1}{1-i} \\ &= \frac{(-2+2i)(1-i) + (3+2i)}{(3+2i)(1-i)} = \frac{3+6i}{5-i} = \frac{(3+6i)(5+i)}{(5-i)(5+i)} \\ &= \frac{9+33i}{26} = \frac{9}{26} + \frac{33}{26}i. \end{aligned}$$

(b) $|w| = \sqrt{\left(\frac{9}{26}\right)^2 + \left(\frac{33}{26}\right)^2} = \frac{\sqrt{1170}}{26} = \frac{3\sqrt{130}}{26}.$

5. We rewrite the equation in the form

$$z^5 = -\frac{1}{2} + \frac{\sqrt{3}i}{2}.$$

Since the modulus of the complex number on the right is $\sqrt{1/4 + 3/4} = 1$, and an argument is $2\pi/3$, we can write that

$$z^5 = e^{2\pi i/3} = e^{(2\pi/3 + 2k\pi)i},$$

where k is an integer. When we take fifth roots,

$$z = e^{(2\pi/3+2k\pi)i/5} = e^{(2\pi/15+2k\pi/5)i}.$$

For $k = 0, 1, 2, 3, 4$, we obtain the roots:

$$\begin{aligned}
z_0 &= e^{2\pi i/15},\\
z_1 &= e^{8\pi i/15},\\
z_2 &= e^{14\pi i/15},\\
z_3 &= e^{20\pi i/15} = e^{-2\pi i/3},\\
z_4 &= e^{26\pi i/15} = e^{-4\pi i/15}.
\end{aligned}$$

6. If we set $z = x + yi$, the equations become

$$|x + yi + 1 + 3i| = \sqrt{34}, \qquad (1 - 3i)(x + yi) + (1 + 3i)(x - yi) = 4.$$

The second equation gives

$$(x - 3xi + yi + 3y) + (x + 3xi - yi + 3y) = 4 \quad\Longrightarrow\quad 2x + 6y = 4 \quad\Longrightarrow\quad x = 2 - 3y.$$

If we square the first equation, and substitute $x = 2 - 3y$,

$$\begin{aligned}
34 &= |(x + 1) + (y + 3)i|^2 = |((2 - 3y + 1) + (y + 3)i|^2\\
&= (3 - 3y)^2 + (y + 3)^2 = 10y^2 - 12y + 18.
\end{aligned}$$

Thus,

$$0 = 10y^2 - 12y - 16 = 2(y - 2)(5y + 4),$$

solutions of which are $y = 2$ and $y = -4/5$. Corresponding values for x are -4 and $22/5$. The complex numbers are

$$-4 + 2i, \qquad \frac{22}{5} - \frac{4i}{5}.$$