MATH 1210 Assignment 3 Solutions Fall 2018

1. Given the vectors $\mathbf{u} = \langle 3, 1\rangle$, $\mathbf{v} = \langle 2, -4\rangle$, and $\mathbf{w} = \langle 5, -2\rangle$, find scalars a and b such that $\mathbf{w} = a\mathbf{u} + b\mathbf{v}$.

If we substitute components for the three vectors, we obtain

$$\langle 5, -2\rangle = a\langle 3, 1\rangle + b\langle 2, -4\rangle.$$

We now equate x- and y-components of these vectors,

$$5 = 3a + 2b, \quad -2 = a - 4b.$$

The solution of these equations is $a = 8/7$ and $b = 11/14$.

2. Show that the lines

$$\begin{aligned} x &= 1 - t, \\ y &= -3 + 3t, \\ z &= 2 + 4t, \end{aligned} \qquad \text{and} \qquad \begin{aligned} x &= 3 + 5s/2, \\ y &= 2 + 7s/2, \\ z &= 5 + s, \end{aligned}$$

intersect, and find the acute angle between them.

When we equate x and y-values, we obtain

$$1 - t = 3 + \frac{5s}{2}, \quad -3 + 3t = 2 + \frac{7s}{2}.$$

The solution of these equations is $t = 1/2$ and $s = -1$. These give $x = 1/2$ and $y = -3/2$. Both equations for z gives $z = 4$ and the point of intersection of the lines is therefore $(1/2, -3/2, 4)$. Vectors along the lines are $\langle -1, 3, 4\rangle$ and $\langle 5, 7, 2\rangle$. If θ is the angle between the lines, we can write that

$$\langle -1, 3, 4\rangle \cdot \langle 5, 7, 2\rangle = |\langle -1, 3, 4\rangle||\langle 5, 7, 2\rangle| \cos\theta.$$

Thus,

$$\cos\theta = \frac{-1(5) + 3(7) + 4(2)}{\sqrt{1 + 9 + 16}\,\sqrt{25 + 49 + 4}} = \frac{24}{\sqrt{26}\,\sqrt{78}} \qquad \Longrightarrow \qquad \theta = \text{Cos}^{-1}\frac{12}{13\sqrt{3}} = 1.01 \text{ radians.}$$

3. Find the components of all vectors $\mathbf{v}$ which have length 2 and are perpendicular to both the lines $x = 4 + 3t$, $y = 2 - t$, $z = 1 + 5t$ and $x - y + z = 2$, $3x + 2y - 4z = 6$.

A vector along the second line is

$$\mathbf{v} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & -1 & 1 \\ 3 & 2 & -4 \end{vmatrix} = \langle 2, 7, 5 \rangle.$$

Since a vector along the first line is $\mathbf{u} = \langle 3, -1, 5 \rangle$, a vector perpendicular to both lines is

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 3 & -1 & 5 \\ 2 & 7 & 5 \end{vmatrix} = \langle -40, -5, 23 \rangle.$$

A unit vector in this direction is

$$\frac{\langle -40, -5, 23 \rangle}{\sqrt{1600 + 25 + 529}} = \frac{\langle -40, -5, 23 \rangle}{\sqrt{2154}}.$$

There are two vectors of length 2 perpendicular to both lines, namely,

$$\pm \frac{2\langle -40, -5, 23 \rangle}{\sqrt{2154}}.$$

4. Find the equation of the plane, simplified as much as possible, that contains the point where the line $x = -1 + 2t$, $y = -4 + 2t$, $z = 1 - 4t$ intersects the xz-plane, and is perpendicular to the line

$$\frac{x+1}{3} = \frac{3y+1}{6} = \frac{1-2z}{4}.$$

To find where the line intersects the xz-plane, we set

$$0 = y = -4 + 2t \qquad \Longrightarrow \qquad t = 2.$$

This gives the point $(3, 0, -7)$. When we write the symmetric equations of the second line in the form

$$\frac{x+1}{3} = \frac{y+1/3}{2} = \frac{z-1/2}{-2},$$

we see that a vector along the line is $\langle 3, 2, -2 \rangle$. Since this vector is normal to the plane, the equation of the plane is

$$3(x-3) + 2(y) - 2(z+7) = 0 \qquad \Longrightarrow \qquad 3x + 2y - 2z = 23.$$

5. Find, if possible, the equation of a plane containing the two lines

$$x - y + 2z = 9, \qquad 2x + y - 4z = -12,$$
$$2x + y - 3z = -9, \quad \text{and} \quad x + 3y + 5z = 10.$$

For the lines to determine a plane, they must either be parallel or intersect. Since vectors along the lines are

$$\mathbf{u} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & -1 & 2 \\ 2 & 1 & -3 \end{vmatrix} = \langle 1, 7, 3 \rangle, \qquad \mathbf{v} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 2 & 1 & -4 \\ 1 & 3 & 5 \end{vmatrix} = \langle 17, -14, 5 \rangle,$$

and these vectors are not multiples of each other, the lines are not parallel. To determine whether the lines intersect, we solve all four equations simultaneously. The augmented matrix is

$$\left(\begin{array}{ccc|c} 1 & -1 & 2 & 9 \\ 2 & 1 & -3 & -9 \\ 2 & 1 & -4 & -12 \\ 1 & 3 & 5 & 10 \end{array}\right) \begin{array}{l} \\ R_2 \to -2R_1 + R_2 \\ R_3 \to -2R_1 + R_3 \\ R_4 \to -R_1 + R_4 \end{array} \longrightarrow \left(\begin{array}{ccc|c} 1 & -1 & 2 & 9 \\ 0 & 3 & -7 & -27 \\ 0 & 3 & -8 & -30 \\ 0 & 4 & 3 & 1 \end{array}\right) \begin{array}{l} \\ \\ \\ R_4 \to -R_2 + R_4 \end{array}$$

$$\longrightarrow \left(\begin{array}{ccc|c} 1 & -1 & 2 & 9 \\ 0 & 3 & -7 & -27 \\ 0 & 3 & -8 & -30 \\ 0 & 1 & 10 & 28 \end{array}\right) \begin{array}{l} \\ R_2 \to R_4 \\ \\ R_4 \to R_2 \end{array} \longrightarrow \left(\begin{array}{ccc|c} 1 & -1 & 2 & 9 \\ 0 & 1 & 10 & 28 \\ 0 & 3 & -8 & -30 \\ 0 & 3 & -7 & -27 \end{array}\right) \begin{array}{l} R_1 \to R_2 + R_1 \\ \\ R_3 \to -3R_2 + R_3 \\ R_4 \to -3R_2 + R_4 \end{array}$$

$$\longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & 12 & 37 \\ 0 & 1 & 10 & 28 \\ 0 & 0 & -38 & -114 \\ 0 & 0 & -37 & -111 \end{array}\right) \begin{array}{l} \\ \\ R_3 \to -R_3/38 \\ R_4 \to -R_4/37 \end{array} \longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & 12 & 37 \\ 0 & 1 & 10 & 28 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 1 & 3 \end{array}\right) \begin{array}{l} R_1 \to -12R_3 + R_1 \\ R_2 \to -10R_3 + R_2 \\ \\ R_4 \to -R_3 + R_4 \end{array}$$

$$\longrightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

The lines therefore intersect at the point $(1, -2, 3)$. A vector normal to the plane is

$$\mathbf{N} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & 7 & 3 \\ 17 & -14 & 5 \end{vmatrix} = \langle 77, 46, -133 \rangle.$$

The equation of the plane is therefore

$$77(x-1) + 46(y+2) - 133(z-3) = 0 \qquad \Longrightarrow \qquad 77x + 46y - 133z = -414.$$

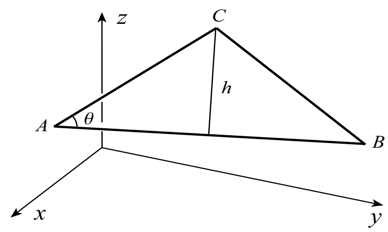
6. (a) Prove that if A, B, and C are three points in space, then the area of triangle ABC can be calculated with the formula

$$\text{Area of } \Delta\, ABC = \frac{1}{2}|\mathbf{AB} \times \mathbf{AC}|.$$

(b) Use the formula in part (a) to find the area of the triangle with vertices $(2, 0, -3)$, $(1, 5, 6)$, and $(-1, 3, 4)$.

(a) From the figure below, we can say that

$$\text{Area of } \Delta\, ABC = \frac{1}{2}|\mathbf{AB}|h = \frac{1}{2}|\mathbf{AB}||\mathbf{AC}|\sin\theta = \frac{1}{2}|\mathbf{AB} \times \mathbf{AC}|.$$



(b) If we denote the vertices by $A(2, 0, -3)$, $B(1, 5, 6)$ and $C(-1, 3, 4)$, then the area of the triangle is

$$\frac{1}{2}|\mathbf{AB} \times \mathbf{AC}| = \frac{1}{2}\left\| \begin{matrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ -1 & 5 & 9 \\ -3 & 3 & 7 \end{matrix} \right\| = \frac{1}{2}|\langle 8, -20, 12 \rangle| = \frac{1}{2}\sqrt{64 + 400 + 144} = \sqrt{151}.$$