MATH 1210 Assignment 4 Solutions Fall 2018

1. State whether each of the following matrices is in row echelon form. If a matrix is not in row echelon form, use elementary row operations to change it to row echelon form. Use the notation of the notes to indicate the elementary row operations used.

(a) $\begin{pmatrix} 1 & 3 & -2 \\ 0 & 1 & 4 \\ 0 & 0 & 2 \end{pmatrix}$ (b) $\begin{pmatrix} 2 & 1 & 3 & 4 \\ 0 & 1 & 0 & 2 \\ 0 & 2 & 1 & 4 \end{pmatrix}$ (c) $\begin{pmatrix} 1 & 0 & 2 \\ 0 & 4 & 3 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$

(a) The matrix is not in REF.

$$\begin{pmatrix} 1 & 3 & -2 \\ 0 & 1 & 4 \\ 0 & 0 & 2 \end{pmatrix} \; R_3 \to R_3/2 \quad \longrightarrow \quad \begin{pmatrix} 1 & 3 & -2 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{pmatrix}$$

(b) The matrix is not in REF.

$$\begin{pmatrix} 2 & 1 & 3 & 4 \\ 0 & 1 & 0 & 2 \\ 0 & 2 & 1 & 4 \end{pmatrix} \; \begin{matrix} R_1 \to R_1/2 \\ \\ R_3 \to -2R_2 + R_3 \end{matrix} \quad \longrightarrow \quad \begin{pmatrix} 1 & 1/2 & 3/2 & 2 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

(c) The matrix is not in REF.

$$\begin{pmatrix} 1 & 0 & 2 \\ 0 & 4 & 3 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \; \begin{matrix} R_2 \to R_4 \\ R_3 \to R_2 \\ R_4 \to R_3 \end{matrix} \quad \longrightarrow \quad \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 4 & 3 \\ 0 & 0 & 0 \end{pmatrix} \; R_3 \to -4R_2 + R_3$$

$$\longrightarrow \quad \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{pmatrix} \; R_3 \to R_3/3 \quad \longrightarrow \quad \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

2. State whether each of the following matrices is in reduced row echelon form. If a matrix is not in reduced row echelon form, use elementary row operations to change it to reduced row echelon form. Use the notation of the notes to indicate the elementary row operations used.

(a) $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 5 & 1 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 3 \end{pmatrix}$ (b) $\begin{pmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 3 \end{pmatrix}$ (c) $\begin{pmatrix} 2 & 0 & 1 & -6 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 3 & 3 \end{pmatrix}$

(a) The matrix is not in RREF.

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 5 & 1 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 3 \end{pmatrix} \begin{matrix} \\ \\ R_3 \to R_3/2 \\ R_4 \to R_4/3 \end{matrix} \longrightarrow \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 5 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{matrix} R_2 \to R_3 \\ R_3 \to R_2 \end{matrix}$$

$$\longrightarrow \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 5 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{matrix} R_1 \to -3R_2 + R_1 \\ \\ R_3 \to -5R_2 + R_3 \\ \end{matrix} \longrightarrow \begin{pmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -4 \\ 0 & 0 & 0 & 1 \end{pmatrix} R_3 \to -R_3/4$$

$$\longrightarrow \begin{pmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{matrix} R_1 \to -R_3 + R_1 \\ R_2 \to -R_3 + R_2 \\ \\ R_4 \to -R_3 + R_4 \end{matrix} \longrightarrow \begin{pmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

(b) The matrix is in RREF.
(c) The matrix is not in RREF.

$$\begin{pmatrix} 2 & 0 & 1 & -6 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 3 & 3 \end{pmatrix} R_3 \to R_3/3 \longrightarrow \begin{pmatrix} 2 & 0 & 1 & -6 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} \begin{matrix} R_1 \to -R_3 + R_1 \\ R_2 \to -2R_3 + R_2 \\ \\ \end{matrix}$$

$$\longrightarrow \begin{pmatrix} 2 & 0 & 0 & -7 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix} \begin{matrix} R_1 \to R_1/2 \\ \\ \\ \end{matrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 & -7/2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

3. Use Gaussian elimination to find all solutions of the system of equations

$$3x + 2y + z = 4,$$
$$2x - y + 4z = 8.$$

$$\left(\begin{array}{ccc|c} 3 & 2 & 1 & 4 \\ 2 & -1 & 4 & 8 \end{array}\right) R_1 \to -R_2 + R_1 \longrightarrow \left(\begin{array}{ccc|c} 1 & 3 & -3 & -4 \\ 2 & -1 & 4 & 8 \end{array}\right) R_2 \to -2R_1 + R_2$$

$$\longrightarrow \left(\begin{array}{ccc|c} 1 & 3 & -3 & -4 \\ 0 & -7 & 10 & 16 \end{array}\right) R_2 \to -R_2/7 \longrightarrow \left(\begin{array}{ccc|c} 1 & 3 & -3 & -4 \\ 0 & 1 & -10/7 & -16/7 \end{array}\right)$$

When we convert back to equations, we obtain

$$x + 3y - 3z = -4, \qquad y - \frac{10z}{7} = -\frac{16}{7}.$$

From the second,

$$y = \frac{10z}{7} - \frac{16}{7}.$$

Substitution of this into the first equation gives

$$x + 3\left(\frac{10z}{7} - \frac{16}{7}\right) - 3z = -4$$

$$\begin{aligned} x &= -\frac{30z}{7} + 3z + \frac{48}{7} - 4 \\ &= -\frac{9z}{7} + \frac{20}{7}. \end{aligned}$$

z is arbitrary.

4. Use Gauss-Jordan elimination to find all solutions of the system of equations

$$\begin{aligned} x + y - z + 2w &= 1, \\ 3x - 2y + 2z - w &= 0, \\ 4x + y - 2z + 2w &= 2, \\ -2x - 2y + 3z + w &= -1. \end{aligned}$$

$$\left(\begin{array}{cccc|c} 1 & 1 & -1 & 2 & 1 \\ 3 & -2 & 2 & -1 & 0 \\ 4 & 1 & -2 & 2 & 2 \\ -2 & -2 & 3 & 1 & -1 \end{array}\right) \begin{array}{c} \\ R_2 \to -3R_1 + R_2 \\ R_3 \to -4R_1 + R_3 \\ R_4 \to 2R_1 + R_4 \end{array} \longrightarrow \left(\begin{array}{cccc|c} 1 & 1 & -1 & 2 & 1 \\ 0 & -5 & 5 & -7 & -3 \\ 0 & -3 & 2 & -6 & -2 \\ 0 & 0 & 1 & 5 & 1 \end{array}\right) R_2 \to -2R_3 + R_2$$

$$\longrightarrow \left(\begin{array}{cccc|c} 1 & 1 & -1 & 2 & 1 \\ 0 & 1 & 1 & 5 & 1 \\ 0 & -3 & 2 & -6 & -2 \\ 0 & 0 & 1 & 5 & 1 \end{array}\right) \begin{array}{c} R_1 \to -R_2 + R_1 \\ \\ R_3 \to 3R_2 + R_3 \\ \\ \end{array} \longrightarrow \left(\begin{array}{cccc|c} 1 & 0 & -2 & -3 & 0 \\ 0 & 1 & 1 & 5 & 1 \\ 0 & 0 & 5 & 9 & 1 \\ 0 & 0 & 1 & 5 & 1 \end{array}\right) \begin{array}{c} \\ \\ R_3 \to R_4 \\ R_4 \to R_3 \end{array}$$

$$\longrightarrow \left(\begin{array}{cccc|c} 1 & 0 & -2 & -3 & 0 \\ 0 & 1 & 1 & 5 & 1 \\ 0 & 0 & 1 & 5 & 1 \\ 0 & 0 & 5 & 9 & 1 \end{array}\right) \begin{array}{c} R_1 \to 2R_3 + R_1 \\ R_2 \to -R_3 + R_2 \\ \\ R_4 \to -5R_3 + R_4 \end{array} \longrightarrow \left(\begin{array}{cccc|c} 1 & 0 & 0 & 7 & 2 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 5 & 1 \\ 0 & 0 & 0 & -16 & -4 \end{array}\right) R_4 \to -R_4/16$$

$$\longrightarrow \left(\begin{array}{cccc|c} 1 & 0 & 0 & 7 & 2 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 5 & 1 \\ 0 & 0 & 0 & 1 & 1/4 \end{array}\right) \begin{array}{c} R_1 \to -7R_4 + R_1 \\ \\ R_3 \to -5R_4 + R_3 \\ \\ \end{array} \longrightarrow \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1/4 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1/4 \\ 0 & 0 & 0 & 1 & 1/4 \end{array}\right)$$

The solution is $x = 1/4$, $y = 0$, $z = -1/4$, and $w = 1/4$.

5. Use Cramer's rule to find the value of z that satisfies the equations

$$\begin{aligned} x + 2y - 3z &= 4, \\ 2x - y + 3z &= -6, \\ 3x + 3y - 3z &= 4. \end{aligned}$$

According to Cramer's rule,

$$z = \frac{\begin{vmatrix} 1 & 2 & 4 \\ 2 & -1 & -6 \\ 3 & 3 & 4 \end{vmatrix}}{\begin{vmatrix} 1 & 2 & -3 \\ 2 & -1 & 3 \\ 3 & 3 & -3 \end{vmatrix}} = \frac{1(14) - 2(26) + 4(9)}{1(-6) - 2(-15) - 3(9)} = \frac{2}{3}.$$

6. Find the value of k so that 33 is the value of the determinant

$$\begin{vmatrix} 3 & 2 & -1 & 4 \\ 2 & k & 6 & 5 \\ -3 & 2 & 1 & 0 \\ 6 & 4 & 2 & 3 \end{vmatrix}.$$

To evaluate the determinant of the matrix, you must reduce the matrix to upper triangular form.

First we evaluate the determinant by reducing it to upper triangular form,

$$\begin{vmatrix} 3 & 2 & -1 & 4 \\ 2 & k & 6 & 5 \\ -3 & 2 & 1 & 0 \\ 6 & 4 & 2 & 3 \end{vmatrix} \begin{matrix} \\ \\ R_3 \to R_1 + R_3 \\ R_4 \to -2R_1 + R_4 \end{matrix} = \begin{vmatrix} 3 & 2 & -1 & 4 \\ 2 & k & 6 & 5 \\ 0 & 4 & 0 & 4 \\ 0 & 0 & 4 & -5 \end{vmatrix} \begin{matrix} C_2 \to C_4 \\ \\ C_4 \to C_2 \end{matrix}$$

$$= -\begin{vmatrix} 3 & 4 & -1 & 2 \\ 2 & 5 & 6 & k \\ 0 & 4 & 0 & 4 \\ 0 & -5 & 4 & 0 \end{vmatrix} \begin{matrix} R_1 \to -R_2 + R_1 \\ \\ \\ \\ \end{matrix} = -4\begin{vmatrix} 1 & -1 & -7 & 2-k \\ 2 & 5 & 6 & k \\ 0 & 1 & 0 & 1 \\ 0 & -5 & 4 & 0 \end{vmatrix} R_2 \to -2R_1 + R_2$$

$$= -4\begin{vmatrix} 1 & -1 & -7 & 2-k \\ 0 & 7 & 20 & 3k-4 \\ 0 & 1 & 0 & 1 \\ 0 & -5 & 4 & 0 \end{vmatrix} \begin{matrix} R_2 \to R_3 \\ R_3 \to R_2 \end{matrix} = 4\begin{vmatrix} 1 & -1 & -7 & 2-k \\ 0 & 1 & 0 & 1 \\ 0 & 7 & 20 & 3k-4 \\ 0 & -5 & 4 & 0 \end{vmatrix} \begin{matrix} \\ \\ R_3 \to -7R_2 + R_3 \\ R_4 \to 5R_2 + R_4 \end{matrix}$$

$$= 4\begin{vmatrix} 1 & -1 & -7 & 2-k \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 20 & 3k-11 \\ 0 & 0 & 4 & 5 \end{vmatrix} \begin{matrix} \\ \\ R_3 \to R_4 \\ R_4 \to R_3 \end{matrix} = -4\begin{vmatrix} 1 & -1 & -7 & 2-k \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 4 & 5 \\ 0 & 0 & 20 & 3k-11 \end{vmatrix} \begin{matrix} \\ \\ \\ R_4 \to -5R_3 + R_4 \end{matrix}$$

$$= -4\begin{vmatrix} 1 & -1 & -7 & 2-k \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 4 & 5 \\ 0 & 0 & 0 & 3k-36 \end{vmatrix}$$

The value of this determinant is the product of the diagonal entries, and we therefore set

$$-4(4)(3k - 36) = 33, \quad \text{from which} \quad k = \frac{543}{48} = \frac{181}{16}.$$

7. (a) For what value(s) of the constant a will the equations

$$\begin{aligned} ax + y + 2z &= 0, \\ 4x + ay - 3z &= 0, \\ 2x + y + az &= 0, \end{aligned}$$

have nontrivial solutions.

(b) Find all solutions corresponding to the integer value for a.

(a) A homogeneous system has an infinity of solutions only when the determinant of its coefficient matrix has value zero. We therefore set

$$0 = \begin{vmatrix} a & 1 & 2 \\ 4 & a & -3 \\ 2 & 1 & a \end{vmatrix} = a(a^2+3) - 1(4a+6) + 2(4-2a) = a^3 - 5a + 2 = (a-2)(a^2+2a-1).$$

Solutions are

$$a = 2, \qquad a = \frac{-2 \pm \sqrt{4+4}}{2} = -1 \pm \sqrt{2}.$$

(b) When $a = 2$, the equations become

$$\begin{aligned} 2x + y + 2z &= 0, \\ 4x + 2y - 3z &= 0, \\ 2x + y + 2z &= 0. \end{aligned}$$

The augmented matrix for the system is

$$\left(\begin{array}{ccc|c} 2 & 1 & 2 & 0 \\ 4 & 2 & -3 & 0 \\ 2 & 1 & 2 & 0 \end{array}\right) \begin{array}{l} R_2 \to -2R_1 + R_2 \\ R_3 \to -R_1 + R_3 \end{array} \longrightarrow \left(\begin{array}{ccc|c} 2 & 1 & 2 & 0 \\ 0 & 0 & -7 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) R_2 \to -R_2/7$$

$$\longrightarrow \left(\begin{array}{ccc|c} 2 & 1 & 2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) R_1 \to -2R_2 + R_1 \longrightarrow \left(\begin{array}{ccc|c} 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) R_1 \to R_1/2$$

$$\longrightarrow \left(\begin{array}{ccc|c} 1 & 1/2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

When we convert to equations, we obtain

$$x + y/2 = 0, \quad z = 0, \quad \Longrightarrow \quad x = -y/2, \quad z = 0 \quad \text{where } y \text{ is arbitrary.}$$