1210 Assignment 2 Solutions

1. Consider the polynomial equation

$$P(x) = 2x^5 + x^4 - 22x^3 + 13x^2 + 20x + 4 = 0.$$

(a) Use Descartes' rules of sign to state the number of possible positive and negative roots of the equation.
(b) Find an upper bound for $|x|$ when x is a root of the equation.
(c) Use the rational root theorem to list possible rational roots of the equation.
(d) Find all roots of the equation.

(a) Since there are two signs changes in the coefficients of $P(x)$, there is either 2 or 0 positive roots of the equation. Since

$$P(-x) = -2x^5 + x^4 + 22x^3 + 13x^2 - 20x + 4$$

has three signs changes, there is 3 or 1 negative root.
(b) Since $M = \max(1, 22, 13, 20, 4) = 22$, an upper bound for $|x|$ is $\frac{22}{2} + 1 = 12$.
(c) Possible rational roots are ± 1, ± 2, ± 4, $\pm\frac{1}{2}$.
(d) By trial and error, $x = 2$ is a root. We factor $x - 2$ from $P(x)$,

$$P(x) = (x - 2)Q(x) = (x - 2)(2x^4 + 5x^3 - 12x^2 - 11x - 2).$$

We find that $x = 2$ is a zero of $Q(x)$ so that

$$P(x) = (x - 2)^2 R(x) = (x - 2)^2(2x^3 + 9x^2 + 6x + 1).$$

We now find that $x = -1/2$ is a zero of $R(x)$, so that

$$P(x) = (x - 2)^2(2x + 1)(x^2 + 4x + 1).$$

The remaining two roots are $x = \dfrac{-4 \pm \sqrt{16 - 4}}{2} = -2 \pm \sqrt{3}$.

2. You are given that $x = 2 - 3i$ is a zero of the polynomial

$$P(x) = 2x^4 - 5x^3 + 21x^2 + 11x + 91.$$

Find all other zeros.

Since $x = 2 - 3i$ is a zero, so also is $x = 2 + 3i$ (since the polynomial is real). This means that $x - 2 + 3i$ and $x - 2 - 3i$ are factors of $P(x)$, and so also is

$$(x-2+3i)(x-2-3i) = x^2 - 4x + 13.$$

When we divide $P(x)$ by this quadratic, we obtain

$$P(x) = (x^2 - 4x + 13)(2x^2 + 3x + 7) = 0.$$

The remaining two zeros are $x = \dfrac{-3 \pm \sqrt{9-56}}{4} = -\dfrac{3}{4} \pm \dfrac{\sqrt{47}i}{4}$.

3. Prove or disprove that for any two $n \times n$ matrices A and B,

$$(A-B)(A+B) = A^2 - B^2.$$

If we expand the product

$$(A-B)(A+B) = A^2 + AB - BA - B^2.$$

Since AB and BA are not always equal, we cannot say that $(A-B)(A+B) = A^2 - B^2$.

4. If $\mathbf{u} = \langle 2, -4, 1\rangle$, $\mathbf{v} = \langle 4, -3, -2\rangle$, and $\mathbf{w} = \langle 4, 1, -5\rangle$, calculate (a) $\mathbf{u}\cdot\mathbf{v}\times\mathbf{w}$, called the scalar triple product, and (b) $\mathbf{u}\times(\mathbf{v}\times\mathbf{w})$, called a vector triple product.

(a)

$$\mathbf{u}\cdot\mathbf{v}\times\mathbf{w} = \mathbf{u}\cdot\begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 4 & -3 & -2 \\ 4 & 1 & -5 \end{vmatrix} = \mathbf{u}\cdot\langle 17, 12, 16\rangle$$
$$= 2(17) - 4(12) + 1(16) = 2$$

(b)

$$\mathbf{u}\times(\mathbf{v}\times\mathbf{w}) = \mathbf{u}\times\begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 4 & -3 & -2 \\ 4 & 1 & -5 \end{vmatrix} = \mathbf{u}\times\langle 17, 12, 16\rangle$$
$$= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 2 & -4 & 1 \\ 17 & 12 & 16 \end{vmatrix} = \langle -76, -15, 92\rangle$$

5. Find all vectors that are perpendicular to $\langle 1, -2, 5\rangle$, have y-components equal to 3 times their x-components, and have length 5.

We let the components of the vector be $\mathbf{v} = \langle a, 3a, c\rangle$. Since $\mathbf{v}$ is perpendicular to $\langle 1, -2, 5\rangle$,

$$0 = \langle 1, -2, 5\rangle \cdot \langle a, 3a, c\rangle = a - 6a + 5c = -5a + 5c.$$

Thus, $c = a$, and $\mathbf{v} = \langle a, 3a, a\rangle$. Since $\mathbf{v}$ must have length 5,

$$5 = \sqrt{a^2 + 9a^2 + a^2} = \sqrt{11a^2} \quad \Longrightarrow \quad 11a^2 = 25 \quad \Longrightarrow \quad a = \pm\frac{5}{\sqrt{11}}.$$

There are two vectors $\pm\dfrac{5}{\sqrt{11}}\langle 1, 3, 1\rangle$.

6. Find parametric and symmetric equations for the line

$$x - y + 2z = 4, \quad 3x + y - z = 7.$$

A vector along the line is

$$\langle 1, -1, 2\rangle \times \langle 3, 1, -1\rangle = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & -1 & 2 \\ 3 & 1 & -1 \end{vmatrix} = \langle -1, 7, 4\rangle.$$

If we set $z = 0$, then

$$x - y = 4, \quad 3x + y = 7.$$

The solution of these is $x = 11/4$ and $y = -5/4$. A point on the line is therefore $(11/4, -5/4, 0)$, and parametric equations for the line are

$$x = \frac{11}{4} - t, \quad y = -\frac{5}{4} + 7t, \quad z = 4t.$$

Symmetric equations are

$$\frac{x - 11/4}{-1} = \frac{y + 5/4}{7} = \frac{z}{4}.$$

In problems 7–8, find out whether there exists a plane containing the two given lines. If there is such a plane, find its equation.

7.

$$L_1: \quad \begin{aligned} x &= 2 - t, \\ y &= 3 + 2t, \\ z &= 4 + t \end{aligned} \qquad\qquad L_2: \quad \begin{aligned} x &= 1 + s, \\ y &= 5 - 2s, \\ z &= 5 + s \end{aligned}$$

Since vectors along the lines are $\langle -1, 2, 1\rangle$ and $\langle 1, -2, 1\rangle$, and they are not multiples of each other, the lines are not parallel. If there is a plane containing the lines, the lines must intersect. If we equate the x's, y's, and z's,

$$2 - t = 1 + s, \quad 3 + 2t = 5 - 2s, \quad 4 + t = 5 + s.$$

The solution of these equations is $t = 1$ and $s = 0$, giving the point of intersection $(1, 5, 5)$. The lines therefore determine a plane. A vector normal to the plane is

$$\langle -1, 2, 1\rangle \times \langle 1, -2, 1\rangle = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ -1 & 2 & 1 \\ 1 & -2 & 1 \end{vmatrix} = \langle 4, 2, 0\rangle.$$

Since $\langle 2, 1, 0\rangle$ is also normal to the plane, the equation of the plane is

$$2(x - 1) + (y - 5) = 0 \qquad \text{or} \qquad 2x + y = 7.$$

8.

$$L_1: \quad \begin{aligned} x &= 1 + t, \\ y &= 2 - t, \\ z &= -3 + 2t \end{aligned} \qquad\qquad L_2: \quad \begin{aligned} x + 2y + z &= 4, \\ x - y + 2z &= -3 \end{aligned}$$

Vectors along the lines are

$$\langle 1, -1, 2\rangle \qquad \text{and} \qquad \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & 2 & 1 \\ 1 & -1 & 2 \end{vmatrix} = \langle 5, -1, -3\rangle.$$

Since these vectors are not multiples, the lines are not parallel. If there is a plane containing the lines, the lines must intersect. If we substitute from the parametric equations into the first plane,

$$4 = (1 + t) + 2(2 - t) + (-3 + 2t) = t + 2.$$

Thus, $t = 2$, and the line intersects the first plane in the point $(3, 0, 1)$. Since this point is not on the second plane, the lines do not intersect. There is no plane.