Midterm Solutions for Math 1210 Fall 2019

1. Use mathematical induction to prove the equality

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots+\frac{1}{n(n+1)}=\frac{n}{n+1}.$$

When $n=1$:

$$\text{L.S.}=\frac{1}{1\cdot 2}=\frac{1}{2} \qquad\qquad \text{R.S.}=\frac{1}{2}.$$

The result is therefore valid for $n=1$. Suppose the result is valid for some integer k; that is,

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots+\frac{1}{k(k+1)}=\frac{k}{k+1}.$$

We must now show that the result is valid for $k+1$; that is,

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots+\frac{1}{(k+1)(k+2)}=\frac{k+1}{k+2}.$$

The left side is equal to

$$\begin{aligned}\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots+\frac{1}{(k+1)(k+2)} &= \left[\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots+\frac{1}{k(k+1)}\right]+\frac{1}{(k+1)(k+2)}\\ &= \frac{k}{k+1}+\frac{1}{(k+1)(k+2)}=\frac{1}{k+1}\left(k+\frac{1}{k+2}\right)\\ &= \frac{1}{k+1}\left(\frac{k^2+2k+1}{k+2}\right)=\frac{1}{k+1}\left[\frac{(k+1)^2}{k+2}\right]=\frac{k+1}{k+2}.\end{aligned}$$

Thus the result is valid for $k+1$, and by mathematical induction, it is valid for all $n\geq 1$.

2. Write the sum

$$\frac{3}{2^7}-\frac{4}{2^8}+\frac{5}{2^9}-\frac{6}{2^{10}}+\cdots+\frac{11}{2^{15}}$$

ine Σ-notation. Your index of summation should start from 1.

The sum can be expressed in the form $\displaystyle\sum_{k=1}^{9}(-1)^{k+1}\frac{k+2}{2^{k+6}}.$

3. Convert the complex number

$$\frac{\overline{1+i}}{\left(\sqrt{2}+e^{\frac{3\pi i}{4}}\right)^6}$$

into Cartesian form.

The denominator is

$$\begin{aligned}\left(\sqrt{2}+e^{\frac{3\pi i}{4}}\right)^6 &= \left(\sqrt{2}+\cos\frac{3\pi}{4}+i\sin\frac{3\pi}{4}\right)^6 = \left(\sqrt{2}-\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}}\right)^6 \\ &= \left(\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}}\right)^6 = \left(e^{\pi i/4}\right)^6 = e^{3\pi i/2} = \cos\frac{3\pi}{2}+i\sin\frac{3\pi}{2} = -i.\end{aligned}$$

Thus,

$$\frac{\overline{1+i}}{\left(\sqrt{2}+e^{\frac{3\pi i}{4}}\right)^6} = \frac{1-i}{-i} = 1+i.$$

6 **4.** Consider the polynomial

$$f(x) = x^3 + ax^2 + bx + c,$$

where a, b, and c are real numbers. It is given that $3-i$ and 2 are roots (=zeros) of $f(x)$. Find the values of a, b, and c.

Since the polynomial has real coefficients, $3+i$ is also a zero. Hence $x-(3+i)$ and $x-(3-i)$ are factors of $f(x)$. When we multiply these together,

$$(x-3-i)(x-3+i) = x^2-6x+10.$$

It now follows that

$$f(x) = x^3+ax^2+bx+c = (x-2)(x^2-6x+10) = x^3-8x^2+22x-20.$$

Thus, $a=-8$, $b=22$, and $c=-20$.

5. Consider the matrix $A = \begin{bmatrix} 3 & -1 \\ 2 & -5 \end{bmatrix}$, and let I_2 denote the 2×2 identity matrix. Now find the matrix

$$A^2+2A-13I_2.$$

$$\begin{aligned}A^2+2A-13I_2 &= \begin{bmatrix} 3 & -1 \\ 2 & -5 \end{bmatrix}\begin{bmatrix} 3 & -1 \\ 2 & -5 \end{bmatrix} + 2\begin{bmatrix} 3 & -1 \\ 2 & -5 \end{bmatrix} - 13\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 7 & 2 \\ -4 & 23 \end{bmatrix} + \begin{bmatrix} -7 & -2 \\ 4 & -23 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.\end{aligned}$$

6. Let I_n denote the $n \times n$ identity matrix. Let A and B be matrices such that the expression

$$B^T I_3 A - 5I_4$$

is defined. Find the sizes of A and B.

Since $-5I_4$ is 4×4, so also must be $B^T I_3 A$. For $B^T I_3$ to be defined, B^T must have 3 columns. We now know that B^T is 4×3, and therefore B is 3×4. For $I_3 A$ to be defined, A must have 3 rows, and therefore A is 3×4.

7. Consider the vectors

$$\mathbf{u} = \langle 4, a, 7 \rangle \qquad \text{and} \qquad \mathbf{v} = \langle a, a-1, -4 \rangle.$$

in $\mathcal{R}^3$. It is given to you that $\mathbf{u}$ and $\mathbf{v}$ are perpendicular (i.e., orthogonal). Find all possible values of a.

Since $\mathbf{u}$ and $\mathbf{v}$ are perpendicular,

$$0 = \mathbf{u} \cdot \mathbf{v} = 4a + a(a-1) - 28 = a^2 + 3a - 28 = (a+7)(a-4).$$

Thus, $a = -7$ or $a = 4$.

8. Consider the lines

$$L_1 : \quad x = 2 - t, \quad y = 3 + t, \quad z = -4 - 6t,$$

and

$$L_2 : \quad x = s, \quad y = 2 - s, \quad z = 3 + 6s,$$

in $\mathcal{R}^3$.
(a) Show that L_1 and L_2 are parallel. (That is to say, give a complete mathematical reasoning.)
(b) Now find the equation of the unique plane which contains both L_1 and L_2.

(a) Vectors along L_1 and L_2 are $\langle -1, 1, -6 \rangle$ and $\langle 1, -1, 6 \rangle$, respectively. Since they are multiples of each other, the vectors are parallel, and therefore the lines are either parallel, or they are the same line. Since the point $(2, 3, -4)$ on L_1 is not on L_2, the lines are different.
(b) Since $(0, 2, 3)$ is a point on L_2, and $(2, 3, -4)$ is a point on L_1, a vector in the required plane is $\langle 2, 3, -4 \rangle - \langle 0, 2, 3 \rangle = \langle 2, 1, -7 \rangle$. A vector normal to the plane is

$$\langle 1, -1, 6 \rangle \times \langle 2, 1, -7 \rangle = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & -1 & 6 \\ 2 & 1 & -7 \end{vmatrix} = \langle 1, 19, 3 \rangle.$$

The equation of the plane is

$$1(x) + 19(y-2) + 3(z-3) = 0 \qquad \Longrightarrow \qquad x + 19y + 3z = 47.$$