Math 1710 Test 2 **April 30, 2017** **50 minutes**

Student Name - **Student Number -**

Values

10 **1.** (a) Show that $\dfrac{d}{dx}\text{Cot}^{-1}x = \dfrac{-1}{1+x^2}$

If we set $y = \text{Cot}^{-1}x$, then $x = \cot y$, and implicit differentiation gives

$$1 = -\csc^2 y \frac{dy}{dx} \qquad \Longrightarrow \qquad \frac{dy}{dx} = -\sin^2 y.$$

Since $\csc^2 y = 1 + \cot^2 y = 1 + x^2$, it follows that $\sin^2 y = 1/(1+x^2)$, and

$$\frac{dy}{dx} = \frac{-1}{1+x^2}.$$

(b) Use your answer from part (a) to evaluate $\displaystyle\int (1+3x^2)\text{Cot}^{-1}x\,dx$.

If we set $u = \text{Cot}^{-1}x$ and $dv = (1+3x^2)\,dx$, then $du = \dfrac{-1}{1+x^2}dx$ and $v = x + x^3$, and integration by parts gives

$$\int (1+3x^2)\text{Cot}^{-1}x\,dx = (x+x^3)\text{Cot}^{-1}x - \int -\frac{x+x^3}{1+x^2}\,dx = (x+x^3)\text{Cot}^{-1}x + \int x\,dx$$
$$= (x+x^3)\text{Cot}^{-1}x + \frac{x^2}{2} + C.$$

7 **2.** Evaluate $\displaystyle\int \frac{x^{2/3}}{1+x^{1/3}}\,dx$.

If we set $u = x^{1/3}$, then $x = u^3$, and $dx = 3u^2\,du$, and

$$\begin{aligned}
\int \frac{x^{2/3}}{1+x^{1/3}}\,dx &= \int \frac{u^2}{1+u}(3u^2)\,du = 3\int \frac{u^4}{1+u}\,du \\
&= 3\int \left(u^3 - u^2 + u - 1 + \frac{1}{1+u}\right)du = 3\left(\frac{u^4}{4} - \frac{u^3}{3} + \frac{u^2}{2} - u + \ln|1+u|\right) + C \\
&= \frac{3}{4}x^{4/3} - x + \frac{3}{2}x^{2/3} - 3x^{1/3} + 3\ln|1+x^{1/3}| + C.
\end{aligned}$$

6 **3.** Evaluate $\displaystyle\int \tan^4(2x)\sec^4(2x)\,dx$.

$$\begin{aligned}
\int \tan^4(2x)\sec^4(2x)\,dx &= \int \tan^4(2x)[1+\tan^2(2x)]\sec^2(2x)\,dx \\
&= \int [\tan^4(2x) + \tan^6(2x)]\sec^2(2x)\,dx \\
&= \frac{1}{10}\tan^5(2x) + \frac{1}{14}\tan^7(2x) + C
\end{aligned}$$

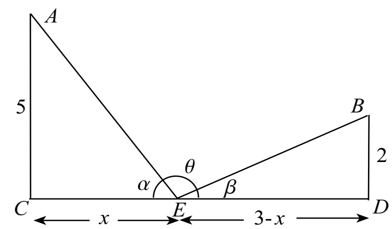
12 **4.** Find the position of the point E along the line CD in the figure below, so that angle AEB is as large as possible.

From the right figure, we can write that

$\theta = \pi - \alpha - \beta$

$= \pi - \mathrm{Cot}^{-1}\left(\frac{x}{5}\right) - \mathrm{Cot}^{-1}\left(\frac{3-x}{2}\right),$

where $0 \le x \le 3$. For critical values of θ, we solve



$$0 = \frac{d\theta}{dx} = \frac{1}{1 + x^2/25}\left(\frac{1}{5}\right) + \frac{1}{1 + (3-x)^2/4}\left(-\frac{1}{2}\right).$$

This can be expressed in the form

$$\frac{5}{x^2 + 25} = \frac{2}{4 + (3-x)^2},$$

from which

$$\begin{aligned} 2x^2 + 50 &= 65 - 30x + 5x^2 \\ 3x^2 - 30x + 15 &= 0 \\ x^2 - 10x + 5 &= 0. \end{aligned}$$

Solutions are $x = \dfrac{10 \pm \sqrt{100 - 20}}{2} = 5 \pm 2\sqrt{5}$, only $x = 5 - 2\sqrt{5}$ being acceptable. We now calculate

$$\begin{aligned} \theta(0) &= \pi - \mathrm{Cot}^{-1}(3/2), \\ \theta(5 - 2\sqrt{5}) &= \pi - \mathrm{Cot}^{-1}\left(\frac{5 - 2\sqrt{5}}{5}\right) - \mathrm{Cot}^{-1}\left(\frac{3 - 5 + 2\sqrt{5}}{2}\right), \\ \theta(3) &= \pi - \mathrm{Cot}^{-1}\left(\frac{3}{5}\right). \end{aligned}$$

The largest of these is the maximum value for θ.