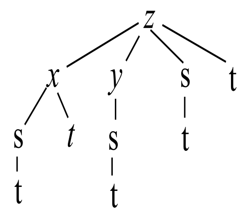
MATH 2130 Test 2 Fall 2024 Solutions

8 **1.** If $z = f(x, y, t, s)$, $x = g(t, s)$, $y = h(s)$, $s = k(t)$, find a formula for dz/dt that involves derivatives of the given functions. Indicate what variables are being held constant in any partial derivatives in the formula.

From the schematic to the right,



$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\bigg)_{y,s,t} \frac{\partial x}{\partial s}\bigg)_t \frac{ds}{dt} + \frac{\partial z}{\partial x}\bigg)_{y,s,t} \frac{\partial x}{\partial t}\bigg)_s + \frac{\partial z}{\partial y}\bigg)_{x,s,t} \frac{dy}{ds}\frac{ds}{dt} + \frac{\partial z}{\partial s}\bigg)_{x,y,t} \frac{ds}{dt} + \frac{\partial z}{\partial t}\bigg)_{x,y,s}.$$

3 **2.** If $f(x, y, z) = x^2 \sin(z/y) + (x^2 + y^2)e^{y^2/(xz)}$, find

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + z\frac{\partial f}{\partial z}.$$

Since $f(tx, ty, tz) = t^2 f(x, y, z)$, the function is positively homogeneous of degree 2, and therefore by Euler's theorem,

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + z\frac{\partial f}{\partial z} = 2f(x, y, z).$$

12 **3.** Find the rate of change of the function $f(x, y, z) = x^2yz - x + y + z$ with respect to distance along the curve

$$x^2y^3 + xyz = 6, \qquad xz^3 + xy + y^3 = 9$$

in the direction of decreasing x at the point $(1, 2, -1)$.

If we set $F(x, y, z) = x^2y^3 + xyz - 6$ and $G(x, y, z) = xz^3 + xy + y^3 - 9$, then

$$\nabla F_{|(1,2,-1)} = (2xy^3 + yz, 3x^2y^2 + xz, xy)_{|(1,2,-1)} = (14, 11, 2),$$
$$\nabla G_{|(1,2,-1)} = (z^3 + y, x + 3y^2, 3xz^2)_{|(1,2,-1)} = (1, 13, 3).$$

A vector tangent to the curve at $(1, 2, -1)$ is

$$\begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 14 & 11 & 2 \\ 1 & 13 & 3 \end{vmatrix} = (7, -40, 171).$$

For decreasing x, we take $\mathbf{T} = (-7, 40, -171)$. Since

$$\nabla f_{|(1,2,-1)} = (2xyz - 1, x^2z + 1, x^2y + 1)_{|(1,2,-1)} = (-5, 0, 3),$$

the required rate of change is

$$D_{\mathbf{T}} f_{|(1,2,-1)} = (-5, 0, 3) \cdot \frac{(-7, 40, -171)}{\sqrt{7^2 + 40^2 + 171^2}} = \frac{-478}{\sqrt{30890}}.$$

12 **4.** The equations

$$u^2 + xv^2 = 2y + xv, \qquad uv^3 + yu = u^3 + x$$

define u and v as functions of x and y. Find $\dfrac{\partial u}{\partial y}$ when $x = 0$, $y = 1$ and u is positive.

If we define $F(x, y, u, v) = u^2 + xv^2 - 2y - xv$ and $G(x, y, u, v) = uv^3 + yu - u^3 - x$, then

$$\frac{\partial u}{\partial y} = -\frac{\frac{\partial(F,G)}{\partial(y,v)}}{\frac{\partial(F,G)}{\partial(u,v)}} = -\frac{\begin{vmatrix} F_y & F_v \\ G_y & G_v \end{vmatrix}}{\begin{vmatrix} F_u & F_v \\ G_u & G_v \end{vmatrix}} = -\frac{\begin{vmatrix} -2 & 2xv - x \\ u & 3uv^2 \end{vmatrix}}{\begin{vmatrix} 2u & 2xv - x \\ v^3 + y - 3u^2 & 3uv^2 \end{vmatrix}}.$$

When $x = 0$ and $y = 1$, the equations yield

$$u^2 = 2, \qquad uv^3 + u = u^3.$$

Since u must be positive, $u = \sqrt{2}$, and $v = 1$. When $x = 0$ and $y = 1$,

$$\frac{\partial u}{\partial y} = -\frac{\begin{vmatrix} -2 & 0 \\ \sqrt{2} & 3\sqrt{2} \end{vmatrix}}{\begin{vmatrix} 2\sqrt{2} & 0 \\ -4 & 3\sqrt{2} \end{vmatrix}} = \frac{1}{\sqrt{2}}.$$

15 **5.** Let $f(x, y) = 2xy^2 + 3xy + x^2y^3$.

(a) Show that $(0, 0)$ is a critical point of the function f and find all other critical points.

(b) Classify the critical point $(0, 0)$ as yielding a relative maximum, a relative minimum, a saddle point, or none of these.

(a) For critical points, we solve

$$0 = f_x = 2y^2 + 3y + 2xy^3 = y(2y + 3 + 2xy^2),$$
$$0 = f_y = 4xy + 3x + 3x^2y^2 = x(4y + 3 + 3xy^2).$$

$(0, 0)$ clearly satisfies these equations, and is therefore a critical point. When we set

$$x = 0, \qquad 2y + 3 + 2xy^2 = 0,$$

we obtain the critical point $(0, -3/2)$. When we set

$$y = 0, \qquad 4y + 3 + 3xy^2 = 0,$$

we obtain no critical points. Finally, we set

$$2y + 3 + 2xy^2 = 0, \qquad 4y + 3 + 3xy^2 = 0.$$

When we add -3 times the first to 2 times the second,

$$-3(2y + 3) + 2(4y + 3) = 0 \qquad \Longrightarrow \qquad y = 3/2.$$

This implies that

$$3 + 3 + 9x/2 = 0 \qquad \Longrightarrow \qquad x = -4/3.$$

Another critical point is therefore $(-4/3, 3/2)$.
Since

$$f_{xx} = 2y^3, \qquad f_{xy} = 4y + 3 + 6xy^2, \qquad f_{yy} = 4x + 6x^2y,$$

we find that at $(0, 0)$, $B^2 - AC = 9$, and therefore $(0, 0)$ yields a saddle point.