## Solutions to Fall 2020 Exam

8 **1.** Set up, **but do NOT evaluate**, a definite integral for the length of the curve

$$x^2y + xz = 2, \quad y - z = 0$$

between the points $(1, 1, 1)$ and $(2, 1/3, 1/3)$.

Let $x(t) = t$. Since $y = z$, we have $y(t) = z(t) = \dfrac{2}{t^2 + t}$. Consequently,

$$\frac{dx}{dt} = 1, \qquad \frac{dy}{dt} = \frac{dz}{dt} = -\frac{2(2t + 1)}{(t^2 + t)^2}.$$

The length of the curve is

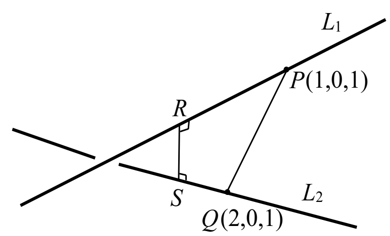
$$L = \int_1^2 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt = \int_1^2 \sqrt{1 + 2\frac{(4t + 2)^2}{(t^2 + t)^4}}\, dt.$$

12 **2.** Find the distance between the lines

$$L_1 : \quad \begin{aligned} x + y &= 1, \\ x - y + 2z &= 3 \end{aligned} \qquad\qquad L_2 : \quad \begin{aligned} x &= 2 + t, \\ y &= 2t, \\ z &= 1 + 2t. \end{aligned}$$

A vector along $L_1$ is

$$\begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & 1 & 0 \\ 1 & -1 & 2 \end{vmatrix} = (2, -2, -2).$$




Since a vector along $L_2$ is $(1, 2, 2)$, the lines are not parallel. A point on $L_1$ is $P(1, 0, 1)$ and a point on $L_2$ is $Q(2, 0, 1)$. A vector along **RS** normal to both lines is

$$\begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & -1 & -1 \\ 1 & 2 & 2 \end{vmatrix} = (0, -3, 3).$$

The distance between the lines is

$$\left|\mathbf{PQ} \cdot \hat{\mathbf{RS}}\right| = \left|(1, 0, 0) \cdot \frac{(0, -3, 3)}{\sqrt{18}}\right| = 0.$$

The lines therefore intersect.

**6** **3.** Determine whether the limit

$$\lim_{(x,y)\to(0,0)} \frac{2xy^2}{x^2+y^4}$$

exists. If the limit does not exist, give reason(s) for its nonexistence.

Along the curves $x = my^2$, we have

$$\lim_{(x,y)\to(0,0)} \frac{2xy^2}{x^2+y^4} = \lim_{y\to 0} \frac{2(my^2)y^2}{(my^2)^2+y^4} = \lim_{y\to 0} \frac{2my^4}{y^4(m^2+1)} = \frac{2m}{m^2+1}.$$

Since these limits depend on $m$, the original limit does not exist.

**11** **4.** Find all critical points for the function

$$f(x,y) = -x^3 + 4xy - 2y^2 + 1,$$

and classify them as giving relative maxima, relative minima, saddle points, or none of these.

From the partial derivatives

$$f_x = -3x^2 + 4y = 0, \qquad f_y = 4x - 4y = 0,$$

we have $x = y$ and $-3x^2 + 4x = 0$. Thus, $x = 0$ and $x = 4/3$, and the critical points are $(0,0)$ and $(4/3, 4/3)$. The second partial derivatives are

$$f_{xx} = -6x, \qquad f_{xy} = 4, \qquad f_{yy} = -4$$

For $(0,0)$, $A = 0$, $B = 4$, and $C = -4$. Hence, $B^2 - AC = 16 > 0$, and $(0,0)$ gives a saddle point. For $(4/3, 4/3)$, $A = -8 < 0$, $B = 4$, and $C = -4$. Hence, $B^2 - AC = -16 < 0$, and $(4/3, 4/3)$ gives a relative maximum.

12 **5.** Let $z = s^2 + t^2$, where $s$ and $t$ are functions of $x$ and $y$ defined by

$$x = t^2 - s^2, \quad y = t^2 - s.$$

Find $\dfrac{\partial z}{\partial x}$ when $s = -1$ and $t = 1$.

The partial derivative is

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial s}\frac{\partial s}{\partial x} + \frac{\partial z}{\partial t}\frac{\partial t}{\partial x}.$$

From the equation for $z$, we have

$$\frac{\partial z}{\partial s} = 2s \quad \Longrightarrow \quad \frac{\partial z}{\partial s}_{|s=-1} = -2, \qquad \frac{\partial z}{\partial t} = 2t \quad \Longrightarrow \quad \frac{\partial z}{\partial t}_{|t=1} = 2.$$

When $s = -1$ and $t = 1$, we have $x = 0$ and $y = 2$. Let $F = x - t^2 + s^2$, and $G = y - t^2 + s$. Then

$$\frac{\partial s}{\partial x} = -\frac{\dfrac{\partial(F,G)}{\partial(x,t)}}{\dfrac{\partial(F,G)}{\partial(s,t)}} = -\frac{\begin{vmatrix} F_x & F_t \\ G_x & G_t \end{vmatrix}}{\begin{vmatrix} F_s & F_t \\ G_s & G_t \end{vmatrix}} = -\frac{\begin{vmatrix} 1 & -2t \\ 0 & -2t \end{vmatrix}}{\begin{vmatrix} 2s & -2t \\ 1 & -2t \end{vmatrix}} = -\frac{\begin{vmatrix} 1 & -2 \\ 0 & -2 \end{vmatrix}}{\begin{vmatrix} -2 & -2 \\ 1 & -2 \end{vmatrix}} = \frac{1}{3},$$

$$\frac{\partial t}{\partial x} = -\frac{\dfrac{\partial(F,G)}{\partial(s,x)}}{\dfrac{\partial(F,G)}{\partial(s,t)}} = -\frac{\begin{vmatrix} F_s & F_x \\ G_s & G_x \end{vmatrix}}{6} = -\frac{\begin{vmatrix} 2s & 1 \\ 1 & 0 \end{vmatrix}}{6} = -\frac{\begin{vmatrix} -2 & 1 \\ 1 & 0 \end{vmatrix}}{6} = \frac{1}{6}.$$

Thus,

$$\frac{\partial z}{\partial x} = (-2)\left(\frac{1}{3}\right) + (2)\left(\frac{1}{6}\right) = -\frac{1}{3}.$$

11 **6.** Find all points on the surface $x^2 + y^2 - z^2 = 1$ at which the normal line to the surface is parallel to the line

$$x = 1 + 3t, \quad y = -2 + 2t, \quad z = 1 - 2t.$$

A normal vector to the surface $f(x, y, z) = x^2 + y^2 - z^2 - 1$ at a point $(a, b, c)$ on the surface is

$$\nabla f_{|(a,b,c)} = (2x, 2y, -2z)_{|(a,b,c)} = (2a, 2b, -2c).$$

So also is $\mathbf{n} = (a, b, -c)$. Since this vector must be parallel to $(3, 2, -2)$, we have $\mathbf{n} = k(3, 2, -2)$ for some constant $k$. Thus,

$$a = 3k, \quad b = 2k, \quad -c = -2k.$$

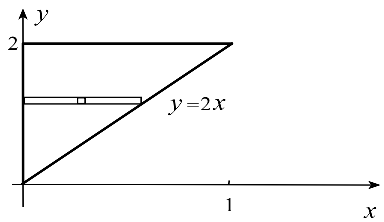
Since point $(a, b, c)$ is on the surface,

$$(3k)^2 + (2k)^2 - (-2k)^2 = 1 \quad \Longrightarrow \quad 9k^2 = 1.$$

Thus, $k = \pm 1/3$. The points are $(1, 2/3, 2/3)$ and $(-1, -2/3, -2/3)$

10 **7.** Evaluate the double integral of the function $e^{y^2}$ over the area bounded by the curves

$$y = 2x, \quad x = 0, \quad y = 2.$$

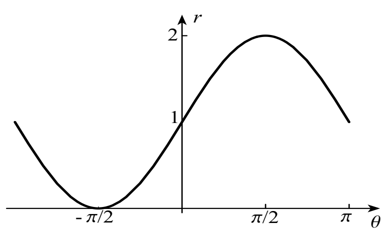


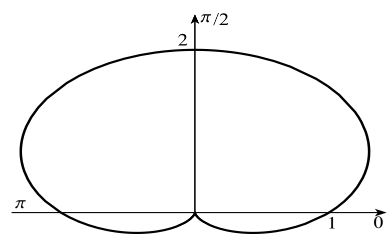

$$\int\int e^{y^2}\,dA = \int_0^2 \int_0^{y/2} e^{y^2}\,dx\,dy = \int_0^2 \left\{xe^{y^2}\right\}_0^{y/2} dy = \int_0^2 \frac{y}{2}e^{y^2}\,dy = \left\{\frac{1}{4}e^{y^2}\right\}_0^2 = \frac{1}{4}(e^4 - 1)$$

9 **8.** Set up, **but do NOT evaluate**, a double iterated integral to find the volume of the solid of revolution when the area bounded by the curve

$$\sqrt{x^2+y^2} = 1 + \frac{y}{\sqrt{x^2+y^2}}$$

is rotated around the line $y = 2$.

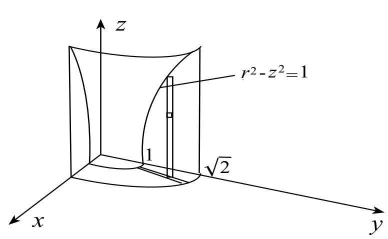



$$V = 2\int_{-\pi/2}^{\pi/2} \int_0^{1+\sin\theta} 2\pi(2-y)r\,dr\,d\theta = 2\int_{-\pi/2}^{\pi/2} \int_0^{1+\sin\theta} 2\pi(2 - r\sin\theta)r\,dr\,d\theta$$

12 **9.** Find the volume bounded by the surfaces

$$x^2 + y^2 - z^2 = 1, \quad x^2 + y^2 = 2.$$

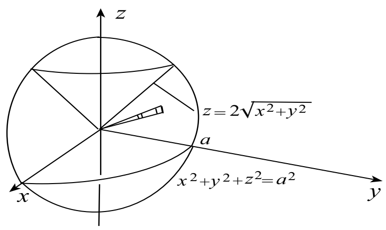



$$V = 8\int_0^{\pi/2} \int_1^{\sqrt{2}} \int_0^{\sqrt{r^2-1}} r\,dz\,dr\,d\theta = 8\int_0^{\pi/2} \int_1^{\sqrt{2}} r\sqrt{r^2-1}\,dr\,d\theta$$

$$= 8\int_0^{\pi/2} \left\{\frac{1}{3}(r^2-1)^{3/2}\right\}_1^{\sqrt{2}} d\theta = \frac{8}{3}\left\{\theta\right\}_0^{\pi/2} = \frac{4\pi}{3}$$

9 **10.** Set up, **but do NOT evaluate**, a triple iterated integral in spherical coordinates to find the larger volume bounded by the surfaces

$$x^2 + y^2 + z^2 = a^2, \quad z = 2\sqrt{x^2 + y^2}, \quad (a > 0 \text{ a constant}).$$




$$V = 4\int_0^{\pi/2}\int_{\mathrm{Tan}^{-1}(1/2)}^{\pi}\int_0^a R^2 \sin\phi \, dR \, d\phi \, d\theta$$