Solutions to 2130 Fall Exam

1. Find parametric equations for that part of the curve

$$2x^2 + y^2 + z^2 = 9, \qquad 4x^2 + y^2 = 4$$

in the first octant to the left of the plane $y = x$, directed so that x decreases along the curve.

If we set $x = \cos t$ and $y = 2\sin t$, then

$$z = \sqrt{9 - 2\cos^2 t - 4\sin^2 t}.$$

When $y = x$,

$$\cos t = 2\sin t \qquad \Longrightarrow \qquad \tan t = \frac{1}{2} \qquad \Longrightarrow \qquad t = \text{Tan}^{-1}(1/2).$$

Values $0 < t < \text{Tan}^{-1}(1/2)$ yield the curve in the correct direction.

2. Find the equation of the plane that contains the lines

$$L_1: \begin{array}{l} x = t + 1, \\ y = 7t - 3, \\ z = 3 + 4t \end{array} \qquad\qquad L_2: \begin{array}{l} x + y - 2z = 0, \\ 3x - y + z = 0 \end{array}$$

A vector along L_1 is $\mathbf{v}_1 = (1, 7, 4)$. A vector along L_2 is

$$\mathbf{v}_2 = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & 1 & -2 \\ 3 & -1 & 1 \end{vmatrix} = (-1, -7, -4).$$

The lines are therefore parallel. Since $P(1, -3, 3)$ is a point on L_1 and $Q(0, 0, 0)$ is a point on L_2, a vector in the required plane is $\mathbf{QP} = (1, -3, 3)$. A normal to the required plane is

$$\begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & 7 & 4 \\ 1 & -3 & 3 \end{vmatrix} = (33, 1, -10).$$

The equation of the plane is

$$33(x - 0) + (y - 0) - 10(x - 0) = 0 \qquad \Longrightarrow \qquad 33x + y - 10z = 0.$$

3. Find the angle between the tangent line to the curve

$$x = t^2 + 3, \qquad y = 3 - t, \qquad z = 4 + t^2$$

and the normal vector to the surface

$$x + y^2 - z = -1$$

at their point of intersection.

To find the point of intersection, we solve

$$(t^2 + 3) + (3 - t)^2 - (4 + t^2) = -1 \quad \Longrightarrow \quad (3 - t)^2 = 0 \quad \Longrightarrow \quad t = 3.$$

The point of intersection is therefore $(12, 0, 13)$. A tangent vector to the curve at this point is

$$\mathbf{T}(3) = (2t, -1, 2t)_{|t=3} = (6, -1, 6).$$

A normal vector to the surface at $(12, 0, 13)$ is

$$\nabla(x + y^2 - z + 1)_{|(12,0,13)} = (1, 2y, -1)_{|(12,0,13)} = (1, 0, -1).$$

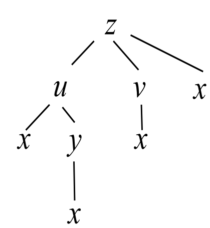
Since $(6, -1, 6) \cdot (1, 0, -1) = 0$, the tangent vector and normal vector are perpendicular. The angle between them is therefore $\pi/2$ radians.

4. You are given that

$$z = f(u, v, x), \quad u = g(x, y), \quad v = h(x), \quad y = k(x).$$

Find an expression for dz/dx in terms of derivatives of the given functions. In all partial derivatives indicate variables that are being held constant. Be sure that your penmenship distinguishes between $d's$ and ∂'s.

From the schematic,



$$\frac{dz}{dx} = \left.\frac{\partial z}{\partial u}\right)_{v,x} \left.\frac{\partial u}{\partial x}\right)_{y} + \left.\frac{\partial z}{\partial u}\right)_{v,x} \left.\frac{\partial u}{\partial y}\right)_{x} \frac{dy}{dx} + \left.\frac{\partial z}{\partial v}\right)_{u,x} \frac{dv}{dx} + \left.\frac{\partial z}{\partial x}\right)_{u,v}.$$

5. In what directions, if any, is the rate of change of the function

$$f(x, y, z) = x^2yz + xze^y$$

at the point $(1, 0, -2)$ equal to -5?

We calculate

$$\nabla f_{|(1,0,-2)} = (2xyz + ze^y, x^2z + xze^y, x^2y + xe^y)_{|(1,0,-2)} = (-2, -4, 1).$$

The minimum rate of increase of the function is $-\sqrt{(-2)^2 + (-4)^2 + 1^2} = -\sqrt{21}$. Thus, there is no direction in which the rate of change is -5.

6. Find the minimum value of the function

$$f(x,y) = 2x^2 + xy - 2x$$

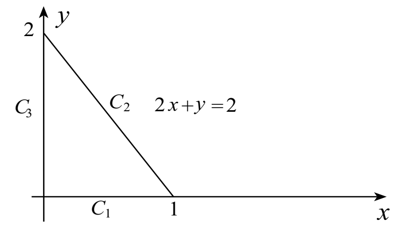
on the region bounded by the lines

$$y = 0, \qquad x = 0, \qquad 2x + y = 2.$$

For critical points inside the region, we solve

$$0 = f_x = 4x + y - 2, \quad 0 = f_y = x.$$

The only critical point is $(0,2)$ at which $f(0,2) = \boxed{0}$. On C_3, where $x = 0$, $f(0,y) = 0$. On C_2, where $y = 2 - 2x$,



$$f(x, 2-2x) = 2x^2 + x(2-2x) - 2x = 0.$$

On C_1, where $y = 0$,

$$g(x) = f(x,0) = 2x^2 - 2x, \quad 0 \le x \le 1.$$

For critical values, we solve

$$0 = g'(x) = 4x - 2 \quad \Longrightarrow \quad x = 1/2.$$

We evaluate

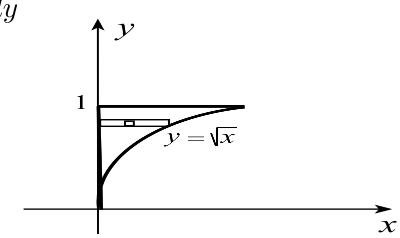
$$g(0) = \boxed{0}, \quad g(1/2) = \boxed{-1/2}, \quad g(1) = \boxed{0}.$$

The minimum value is therefore $-1/2$.

7. Evaluate the double integral of the function $f(x,y) = \sqrt{1-y^3}$ over the region bounded by the curves

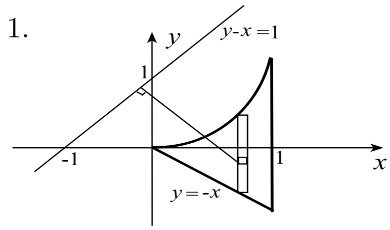
$$y = \sqrt{x}, \qquad y = 1, \qquad x = 0.$$

$$\begin{aligned}\iint_R \sqrt{1-y^3}\,dA &= \int_0^1 \int_0^{y^2} \sqrt{1-y^3}\,dx\,dy \\ &= \int_0^1 y^2\sqrt{1-y^3}\,dy \\ &= \left\{-\frac{2}{9}(1-y^3)^{3/2}\right\}_0^1 \\ &= \frac{2}{9}\end{aligned}$$



8. Set up, **but do NOT evaluate**, a double iterated integral to find the first moment about the line $y - x = 1$ of a mass of density $\rho(x, y) = x^2 + y$ bounded by the curves

$$y = x^2, \qquad y = -x, \qquad x = 1.$$



Simplify the integrand as much as possible.

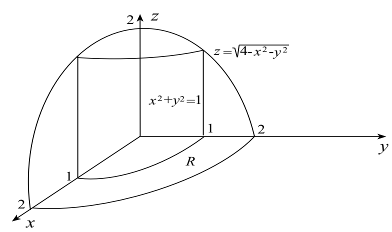
Suppose we choose distance to the right of the line as positive. Then, the moment is

$$\int_0^1 \int_{-x}^{x^2} (x^2 + y) \left(\frac{|y - x - 1|}{\sqrt{2}} \right) dy\, dx = \int_0^1 \int_{-x}^{x^2} (x^2 + y) \left(\frac{x - y + 1}{\sqrt{2}} \right) dy\, dx.$$

9. Set up, **but do NOT evaluate**, a double iterated integral in polar coordinates to find the surface area of the sphere $x^2 + y^2 + z^2 = 4$ that lies outside the cylinder $x^2 + y^2 = 1$.

If R is the area in the first quadrant onto which the surface of the sphere projects, then

$$A = 8 \iint_R \sqrt{1 + \left(\frac{\partial z}{\partial x} \right)^2 + \left(\frac{\partial z}{\partial y} \right)^2}\, dA$$
$$= 8 \iint_R \sqrt{1 + \left(\frac{-x}{\sqrt{4 - x^2 - y^2}} \right)^2 + \left(\frac{-y}{\sqrt{4 - x^2 - y^2}} \right)^2}\, dA$$
$$= 8 \iint_R \sqrt{\frac{4}{4 - x^2 - y^2}}\, dA = 8 \int_0^{\pi/2} \int_1^2 \sqrt{\frac{4}{4 - r^2}}\, r\, dr\, d\theta.$$



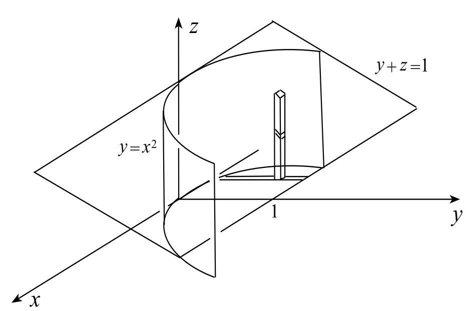
10. Evaluate the triple integral

$$\iiint_W y\, dV$$

where the region W is bounded by

$$y = x^2, \qquad y + z = 1, \qquad z = 0.$$

$$\iiint_W y\, dV = \int_{-1}^1 \int_{x^2}^1 \int_0^{1-y} y\, dz\, dy\, dx$$
$$= \int_{-1}^1 \int_{x^2}^1 y(1 - y)\, dy\, dx$$
$$= \int_{-1}^1 \left\{ \frac{y^2}{2} - \frac{y^3}{3} \right\}_{x^2}^1 dx$$
$$= \int_{-1}^1 \left(\frac{1}{6} - \frac{x^4}{2} + \frac{x^6}{3} \right) dx$$
$$= \left\{ \frac{x}{6} - \frac{x^5}{10} + \frac{x^7}{21} \right\}_{-1}^1 = \frac{8}{35}$$

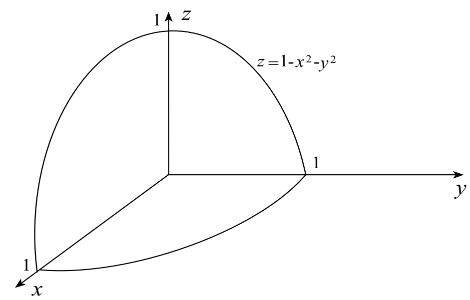


10 **11.** Express, **but do NOT evaluate**, the triple iterated integral

$$\int_{-1}^{1}\int_{0}^{\sqrt{1-x^2}}\int_{0}^{1-x^2-y^2} z\,dz\,dy\,dx$$

in (a) cylindrical coordinates, (b) spherical coordinates.

Because of the symmetry of the volume about the yz-plane, we can double the integral over the volume in the first octant.



(a) If denote the integral by I, then in cylindrical coordinates,

$$I = 2\int_{0}^{\pi/2}\int_{0}^{1}\int_{0}^{1-r^2} z\,r\,dz\,dr\,d\theta$$

(b) In spherical coordinates, the equation of the paraboloid is

$$\mathcal{R}\cos\phi = 1 - \mathcal{R}^2\sin^2\phi \quad\Longrightarrow\quad \mathcal{R}^2\sin^2\phi + \mathcal{R}\cos\phi - 1 = 0.$$

When we solve for $\mathcal{R}$,

$$\mathcal{R} = \frac{-\cos\phi \pm \sqrt{\cos^2\phi + 4\sin^2\phi}}{2\sin^2\phi}.$$

For $\mathcal{R}$ to be positive, we must choose the plus sign, in which case

$$I = 2\int_{0}^{\pi/2}\int_{0}^{\pi/2}\int_{0}^{(-\cos\phi+\sqrt{\cos^2\phi+4\sin^2\phi})/(2\sin^2\phi)} (\mathcal{R}\cos\phi)\mathcal{R}^2\sin\phi\,d\mathcal{R}\,d\phi\,d\theta.$$