2130 Exam Solutions for Fall 2022

1. Set up, **but do NOT evaluate**, a definite integral for the length of the curve

$$x^2 + y + z = 6, \quad 2x - y + z = 8$$

between the points $(2, -1, 3)$ and $(-2, -5, 7)$. Simplify the integrand as much as possible.

If we set $x = t$, then

$$t^2 + y + z = 6, \qquad 2t - y + z = 8.$$

When we solve for y and z, parametric equations of the curve are

$$x = t, \quad y = -1 + t - t^2/2, \quad z = 7 - t - t^2/2, \quad -2 \le t \le 2.$$

The length of the curve is

$$L = \int_{-2}^{2} \sqrt{1 + (1-t)^2 + (-1-t)^2}\, dt = \int_{-2}^{2} \sqrt{3 + 2t^2}\, dt.$$

2. Find the unit tangent vector to the curve

$$x = -t^3, \quad y = 2t^2, \quad z = 3t^3 - t^2$$

at the point $(0, 0, 0)$.

A tangent vector to the curve is

$$\mathbf{T} = (-3t^2, 4t, 9t^2 - 2t) = t(-3t, 4, 9t - 2).$$

So also is the vector $(-3t, 4, 9t - 2)$. At $t = 0$, a tangent vector is $(0, 4, -2)$, and therefore the unit tangent vector is $\dfrac{(0, 2, -1)}{\sqrt{5}}$.

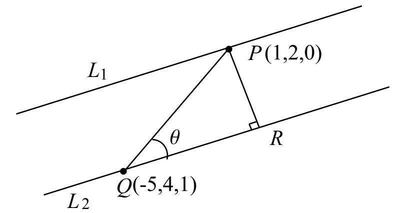
3. (a) Determine whether the following lines are parallel or skew

$$L_1 : x = 1 + t, \quad y = 2 - t, \quad z = 2t \qquad L_2 : x + 5 = 4 - y = \frac{z - 1}{2}$$

(b) Find the distance between the lines.

(a) Since a vector along each line is $(1, -1, 2)$, the lines are parallel.
(b) The distance $|\mathbf{PR}|$ between the lines is given by

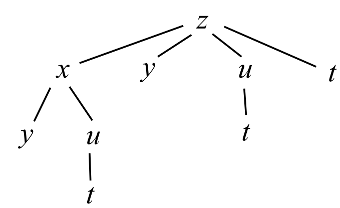
$$|\mathbf{PR}| = |\mathbf{PQ}| \sin\theta = |\mathbf{PQ}||\widehat{\mathbf{QR}}| \sin\theta$$
$$= |\mathbf{PQ} \times \widehat{\mathbf{QR}}| = \frac{1}{\sqrt{6}} \left\| \begin{matrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ -6 & 2 & 1 \\ 1 & -1 & 2 \end{matrix} \right\|$$
$$= \frac{1}{\sqrt{6}} |(5, 13, 4)| = \sqrt{35}.$$



4. Find an expression for $\frac{\partial z}{\partial t}$ if $z = f(x, y, u, t)$, $x = g(y, u)$, and $u = h(t)$. In each partial derivative indicate which variable(s) are held constant.

From the schematic to the right,

$$\frac{\partial z}{\partial t} = \left.\frac{\partial z}{\partial x}\right)_{y,u,t} \left.\frac{\partial x}{\partial u}\right)_{y} \frac{du}{dt} + \left.\frac{\partial z}{\partial u}\right)_{x,y,t} \frac{du}{dt} + \left.\frac{\partial z}{\partial t}\right)_{x,y,u}.$$



5. (a) Find parametric equations for the line normal to the surface $z = 6 - x^2 - y^2$ at the point $(1, 2, 1)$.

(b) Find the points on the line in part (a) $\sqrt{21}$ units away from the surface.

(a) A vector normal to the surface at the point $(1, 2, 1)$ is

$$\nabla(x^2 + y^2 + z - 6)_{|(1,2,1)} = (2x, 2y, 1)_{|(1,2,1)} = (2, 4, 1).$$

Parametric equations for the normal line are therefore

$$x = 1 + 2t, \quad y = 2 + 4t, \quad z = 1 + t.$$

(b) Since the distance between a point (x, y, z) on the normal line and $(1, 2, 1)$ is

$$\sqrt{(x-1)^2 + (y-2)^2 + (z-1)^2} = \sqrt{(1+2t-1)^2 + (2+4t-2)^2 + (1+t-1)^2} = \sqrt{21t^2} = \sqrt{21}|t|,$$

this will be $\sqrt{21}$ when $t = \pm 1$. These values of t give the points $(3, 6, 2)$ and $(-1, -2, 0)$.

6. Find the maximum value of the function $f(x,y) = xy(1-x^2-y)$ on the region bounded by the curves

$$y = 1-x^2, \quad y = 0.$$

For critical points inside the region, we solve

$$0 = f_x = y(1-x^2-y) - 2x(xy) = y(1-y-3x^2),$$
$$0 = f_y = x(1-x^2-y) - xy = x(1-x^2-2y).$$

Obvious solutions are $(0,0)$, $(0,1)$, and $(\pm 1, 0)$. When we set

$$1-y-3x^2 = 0, \quad 1-x^2-2y = 0,$$

we get

$$0 = 1-x^2-2(1-3x^2) \implies 5x^2 = 1 \implies x = \pm\frac{1}{\sqrt{5}}.$$

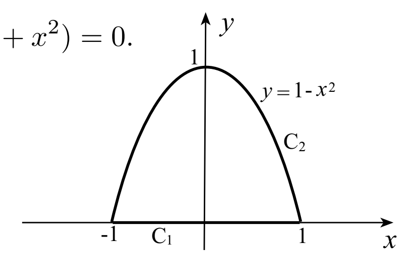
Two additional critical points are therefore $(\pm 1/\sqrt{5}, 2/5)$. We evaluate

$$f(0,0) = \boxed{0}, \quad f(0,1) = \boxed{0}, \quad f(\pm 1, 0) = \boxed{0}, \quad f(\pm 1/\sqrt{5}, 2/5) = \boxed{\pm 4\sqrt{5}/125}.$$

On edge C_1, $f(x,0) = 0$. On edge C_2,

$$f(x, 1-x^2) = x(1-x^2)(1-x^2-1+x^2) = 0.$$

The maximum value of the function is therefore $4\sqrt{5}/125$.

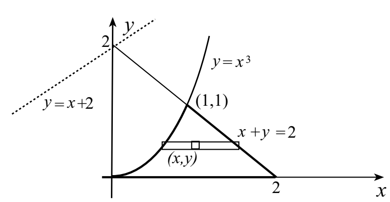


7. Set up, **but do NOT evaluate**, a double iterated integral to find the volume of the solid of revolution when the area bounded by the curves

$$y = x^3, \quad x+y = 2, \quad y = 0$$

is rotated around the line $y = x+2$. Simplify the integrand as much as possible.

$$V = \int_0^1 \int_{y^{1/3}}^{2-y} 2\pi \frac{|y-x-2|}{\sqrt{2}} \, dx\, dy$$
$$= \sqrt{2}\pi \int_0^1 \int_{y^{1/3}}^{2-y} (x+2-y)\, dx\, dy$$

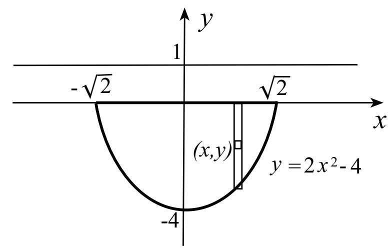


8. Set up, **but do NOT evaluate**, a double iterated integral to find the force due to oil pressure on one side of a plate bounded by the curves

$$y = 2x^2 - 4, \quad y = 0$$

when the plate is submerged vertically in oil with density 915 kilograms per cubic metre if its top edge is one metre below the surface of the oil. (x and y are measured in metres.)

$$F = 2\int_0^{\sqrt{2}} \int_{2x^2-4}^{0} 915(9.81)(1-y)\, dy\, dx \quad \text{N}$$

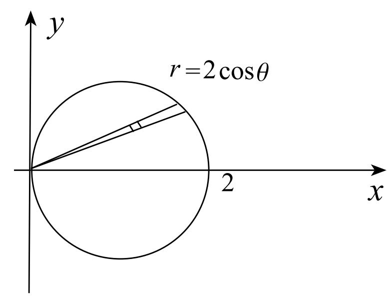


9. Evaluate the double integral

$$\iint_R \sqrt{x^2+y^2}\, dA$$

where R is the region bounded by the curve $x^2 + y^2 = 2x$.

$$\begin{aligned} I &= \iint_R \sqrt{x^2+y^2}\, dA = 2\int_0^{\pi/2} \int_0^{2\cos\theta} (r) r\, dr\, d\theta \\ &= 2\int_0^{\pi/2} \left\{ \frac{r^3}{3} \right\}_0^{2\cos\theta} d\theta = \frac{2}{3}\int_0^{\pi/2} 8\cos^3\theta\, d\theta \\ &= \frac{16}{3}\int_0^{\pi/2} \cos\theta(1-\sin^2\theta)\, d\theta \\ &= \frac{16}{3}\left\{ \sin\theta - \frac{1}{3}\sin^3\theta \right\}_0^{\pi/2} = \frac{32}{9} \end{aligned}$$



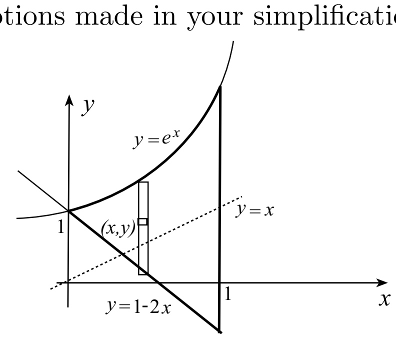
10. Set up, **but do NOT evaluate**, a double iterated integral to find the first moment of a plate with constant mass per unit area ρ about the line $y = x$ if its edges are defined by the curves

$$y = e^x, \quad y = 1 - 2x, \quad x = 1.$$

Simplify the integrand as much as possible. State any assumptions made in your simplification.

If we assume distances to the right of the line are positive, then

$$\begin{aligned} \text{Moment} &= \int_0^1 \int_{1-2x}^{e^x} \frac{|y-x|}{\sqrt{2}} \rho\, dy\, dx \\ &= \frac{\rho}{\sqrt{2}} \int_0^1 \int_{1-2x}^{e^x} (x-y)\, dy\, dx. \end{aligned}$$

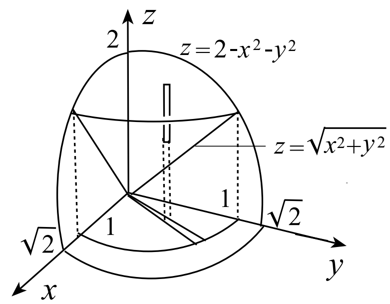


11. Set up, **but do NOT evaluate**, a triple iterated integral in cylindrical coordinates to evaluate the triple integral of the function $f(x,y,z) = z(x^2+y^2)$ over the region V bounded by the surfaces

$$z = \sqrt{x^2+y^2}, \quad z = 2 - x^2 - y^2.$$

Simplify the integrand as much as possible.

$$\iiint_V z(x^2+y^2)\,dV = 4\int_0^{\pi/2}\int_0^1\int_r^{2-r^2} z(r^2)r\,dz\,dr\,d\theta$$
$$= 4\int_0^{\pi/2}\int_0^1\int_r^{2-r^2} r^3\,z\,dz\,dr\,d\theta$$



12. Evaluate the triple integral

$$\iiint_V x\,dV$$

where V is the volume bounded by the surfaces

$$x+y+z=1, \quad x=0, \quad y=0, \quad z=0.$$

$$\iiint_V x\,dV = \int_0^1\int_0^{1-x}\int_0^{1-x-y} x\,dz\,dy\,dx$$
$$= \int_0^1\int_0^{1-x} x(1-x-y)\,dy\,dx$$
$$= \int_0^1 \left\{-\frac{x}{2}(1-x-y)^2\right\}_0^{1-x} dx$$
$$= \frac{1}{2}\int_0^1 x(1-x)^2\,dx$$
$$= \frac{1}{2}\int_0^1 (x-2x^2+x^3)\,dx$$
$$= \frac{1}{2}\left\{\frac{x^2}{2} - \frac{2x^3}{3} + \frac{x^4}{4}\right\}_0^1 = \frac{1}{24}$$