2130 Exam Solutions for Fall 2023

1. Find an equation for the tangent plane to the surface $x^2 + xy + z = 4$ at the point of intersection of the surface with the curve

$$x = t - 1, \quad y = -t - 1, \quad z = t^2 + 3.$$

For the point of intersection, we solve

$$(t-1)^2 + (t-1)(-t-1) + (t^2+3) = 4 \quad \Longrightarrow \quad 0 = t^2 - 2t + 1 = (t-1)^2 \quad \Longrightarrow \quad t = 1.$$

This gives the point $(0, -2, 4)$. A normal vector to the surface at this point is

$$\nabla(x^2 + xy + z - 4)_{|(0,-2,4)} = (2x + y, x, 1)_{|(0,-2,4)} = (-2, 0, 1).$$

The equation of the tangent plane is therefore

$$-2(x - 0) + (z - 4) = 0 \quad \text{or} \quad 2x - z = -4.$$

2. Find the rate of change of the function $f(x, y, z) = x^2 + 3y^2 + yz$ at the point where the line

$$x = t - 2, \quad y = 3 - 4t, \quad z = 6 + 2t$$

intersects the xy-plane in the direction along the line when z is decreasing.

The line intersects the xy-plane when $0 = z = 6 + 2t$ which implies that $t = -3$. The point of intersection is therefore $(-5, 15, 0)$. A tangent vector to the line is $(1, -4, 2)$. We therefore take $\mathbf{T} = (-1, 4, -2)$. Since

$$\nabla f_{|(-5,15,0)} = (2x, 6y + z, y)_{|(-5,15,0)} = (-10, 90, 15),$$

the required rate of change is

$$D_{\mathbf{T}} f = (-10, 90, 15) \cdot \frac{(-1, 4, -2)}{\sqrt{21}} = \frac{340}{\sqrt{21}}.$$

3. Suppose that $z = e^y \sin x$, where x and y are functions of t defined by

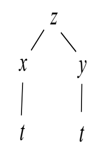
$$xt^2 + x^2t - t + x = 0, \qquad y^3 + e^y - t^2 - t = 1.$$

Find $\dfrac{dz}{dt}$.

The schematic to the right gives

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

$$= (e^y \cos x)\frac{dx}{dt} + (e^y \sin x)\frac{dy}{dt}.$$



If we define $F(x,t) = xt^2 + x^2t - t + x$ and $G(y,t) = y^3 + e^y - t^2 - t - 1$, then

$$\frac{dx}{dt} = -\frac{\frac{\partial(F)}{\partial(t)}}{\frac{\partial(F)}{\partial(x)}} = -\frac{F_t}{F_x} = -\frac{2xt + x^2 - 1}{t^2 + 2xt + 1}, \qquad \frac{dy}{dt} = -\frac{\frac{\partial(G)}{\partial(t)}}{\frac{\partial(G)}{\partial(y)}} = -\frac{G_t}{G_y} = -\frac{-2t-1}{3y^2 + e^y}.$$

Thus,

$$\frac{dz}{dt} = -(e^y \cos x)\left(\frac{2xt + x^2 - 1}{t^2 + 2xt + 1}\right) + (e^y \sin x)\left(\frac{2t+1}{3y^2 + e^y}\right).$$

Alternatively,

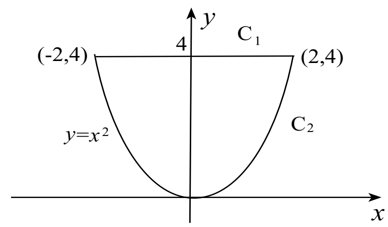
$$\frac{dx}{dt} = -\frac{\frac{\partial(F,G)}{\partial(t,y)}}{\frac{\partial(F,G)}{\partial(x,y)}} = -\frac{\begin{vmatrix} F_t & F_y \\ G_t & G_y \end{vmatrix}}{\begin{vmatrix} F_x & F_y \\ G_x & G_y \end{vmatrix}} = -\frac{\begin{vmatrix} 2xt + x^2 - 1 & 0 \\ -2t-1 & 3y^2 + e^y \end{vmatrix}}{\begin{vmatrix} t^2 + 2xt + 1 & 0 \\ 0 & 3y^2 + e^y \end{vmatrix}}$$

$$= -\frac{(2xt + x^2 - 1)(3y^2 + e^y)}{(t^2 + 2xt + 1)(3y^2 + e^y)} = -\frac{2xt + x^2 - 1}{t^2 + 2xt + 1},$$

$$\frac{dy}{dt} = -\frac{\frac{\partial(F,G)}{\partial(x,t)}}{\frac{\partial(F,G)}{\partial(x,y)}} = -\frac{\begin{vmatrix} F_x & F_t \\ G_x & G_t \end{vmatrix}}{\begin{vmatrix} F_x & F_y \\ G_x & G_y \end{vmatrix}} = -\frac{\begin{vmatrix} t^2 + 2xt + 1 & 2xt + x^2 - 1 \\ 0 & -2t-1 \end{vmatrix}}{\begin{vmatrix} t^2 + 2xt + 1 & 0 \\ 0 & 3y^2 + e^y \end{vmatrix}}$$

$$= -\frac{(t^2 + 2xt + 1)(-2t-1)}{(t^2 + 2xt + 1)(3y^2 + e^y)} = \frac{2t+1}{3y^2 + e^y}.$$

4. Find the absolute maximum and minimum values of the function $f(x, y) = 1 + xy - x - y$ on the region D bounded by the parabola $y = x^2$ and the line $y = 4$.



For critical points inside D, we solve

$$0 = \frac{\partial f}{\partial x} = y - 1, \quad 0 = \frac{\partial f}{\partial y} = x - 1.$$

The solution $(1, 1)$ is on the parabola, and $f(1, 1) = \boxed{0}$.

Along C_1,

$$g(x) = f(x, 4) = 1 + 4x - x - 4 = 3x - 3, \quad -2 \le x \le 2.$$

For critical values of $g(x)$, we solve $0 = g'(x) = 3$. Since there are no critical values, we evaluate $g(-2) = \boxed{-9}$ and $g(2) = \boxed{3}$. Along C_2,

$$h(x) = f(x, x^2) = 1 + x(x^2) - x - x^2 = x^3 - x^2 - x + 1, \quad -2 \le x \le 2.$$

For critical values of $h(x)$, we solve

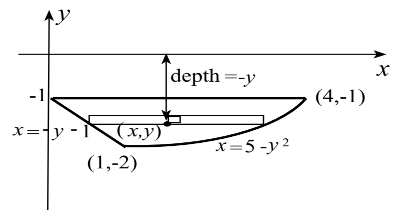
$$0 = h'(x) = 3x^2 - 2x - 1 = (x - 1)(3x + 1) \implies x = 1, \quad x = -1/3.$$

We evaluate

$$h(-2) = \boxed{-9}, \quad h(-1/3) = \boxed{32/27}, \quad h(1) = \boxed{0}, \quad h(2) = \boxed{3}.$$

The absolute maximum is 3, and the absolute minimum is -9.

5. A surface bounded by the parabola $x = 5 - y^2$, the line $x + y + 1 = 0$ and the line $y = -1$ is submerged vertically in water such that the x-axis is on the surface of the water (figure below). Find the force due to water pressure on one side of the surface.



To find the point of intersection of $x = 5 - y^2$ and $x + y + 1 = 0$, we set

$-y - 1 = 5 - y^2 \quad \Longrightarrow \quad 0 = y^2 - y - 6 = (y-3)(y+2).$

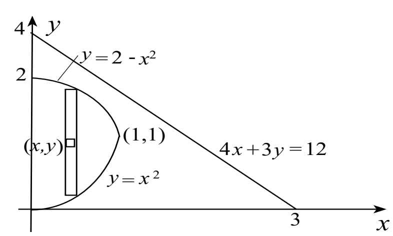
Thus, $(1, -2)$ is the point of intersection.
The force on one side of the surface is

$$F = \int_{-2}^{-1} \int_{-y-1}^{5-y^2} \rho g(-y)\,dx\,dy = -\rho g \int_{-2}^{-1} \{xy\}_{-y-1}^{5-y^2}\,dy$$
$$= -\rho g \int_{-2}^{-1} y[(5 - y^2) - (-y-1)]\,dy = -\rho g \int_{-2}^{-1} (-y^3 + y^2 + 6y)\,dy$$
$$= -\rho g \left\{ -\frac{y^4}{4} + \frac{y^3}{3} + 3y^2 \right\}_{-2}^{-1} = \frac{35\rho g}{12}.$$

6. **Set up but do not evaluate** a double iterated integral for the volume of the solid of revolution when the area bounded by the curves

$$y = x^2, \quad x = \sqrt{2-y}, \quad x = 0$$

is rotated around the line $4x + 3y = 12$. Simplify the integrand as much as possible.

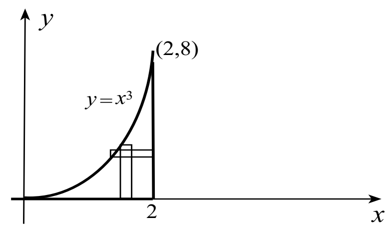


The curves intersect when $x^2 = 2 - x^2$, from which $0 = 2x^2 - 2 = 2(x-1)(x+1)$.
The point of intersection is therefore $(1, 1)$.
The volume is

$$V = \int_0^1 \int_{x^2}^{2-x^2} 2\pi \frac{|4x + 3y - 12|}{5}\,dy\,dx$$
$$= \frac{2\pi}{5} \int_0^1 \int_{x^2}^{2-x^2} (12 - 4x - 3y)\,dy\,dx.$$

7. Evaluate the following double iterated integral.

$$\int_0^8 \int_{\sqrt[3]{y}}^2 \sqrt{x^3+y}\,dx\,dy.$$



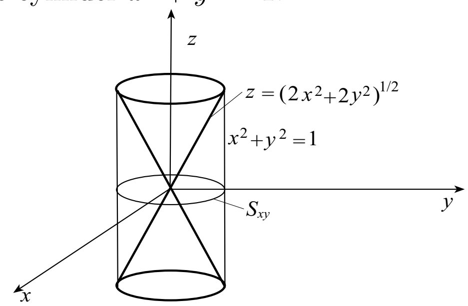
Limits on the integrals indicate that the region of integration is that shown to the right. When we reverse the order of integration,

$$\int_0^8 \int_{\sqrt[3]{y}}^2 \sqrt{x^3+y}\,dx\,dy = \int_0^2 \int_0^{x^3} \sqrt{x^3+y}\,dy\,dx = \int_0^2 \left\{\frac{2}{3}(x^3+y)^{3/2}\right\}_0^{x^3} dx$$

$$= \frac{2}{3}\int_0^2 [(2x^3)^{3/2} - (x^3)^{3/2}]\,dx = \frac{2}{3}(2\sqrt{2}-1)\int_0^2 x^{9/2}\,dx$$

$$= \frac{2}{3}(2\sqrt{2}-1)\left\{\frac{2}{11}x^{11/2}\right\}_0^2 = \frac{4}{33}(2\sqrt{2}-1)2^{11/2} = \frac{128}{33}(4-\sqrt{2}).$$

8. Find the surface area of that part of $z^2 = 2x^2 + 2y^2$ inside the cylinder $x^2 + y^2 = 1$.



If S_1 is that part of the surface above the xy-plane, then Area(S)=2Area(S_1). Thus,

$$\text{Area}(S) = 2\iint_{S_{xy}} \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA$$

where S_{xy} is the area inside $x^2 + y^2 = 1$ in the xy-plane. If we set $F(x, y, z) = z^2 - 2x^2 - 2y^2$, then

$$\frac{\partial z}{\partial x} = -\frac{\frac{\partial(F)}{\partial(x)}}{\frac{\partial(F)}{\partial(z)}} = -\frac{F_x}{F_z} = -\frac{-4x}{2z} = \frac{2x}{z}, \quad \frac{\partial z}{\partial y} = -\frac{\frac{\partial(F)}{\partial(y)}}{\frac{\partial(F)}{\partial(z)}} = -\frac{F_y}{F_z} = -\frac{-4y}{2z} = \frac{2y}{z}.$$

Thus,

$$\sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} = \sqrt{1 + \frac{4x^2}{z^2} + \frac{4y^2}{z^2}} = \sqrt{\frac{z^2 + 4x^2 + 4y^2}{z^2}} = \sqrt{\frac{z^2 + 2z^2}{z^2}} = \sqrt{3}.$$

Alternatively, if we solve for $z = \sqrt{2}\sqrt{x^2 + y^2}$, then

$$\frac{\partial z}{\partial x} = \frac{\sqrt{2}x}{\sqrt{x^2 + y^2}}, \qquad \frac{\partial z}{\partial y} = \frac{\sqrt{2}y}{\sqrt{x^2 + y^2}},$$

and

$$\sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} = \sqrt{1 + \left(\frac{\sqrt{2}x}{\sqrt{x^2 + y^2}}\right)^2 + \left(\frac{\sqrt{2}y}{\sqrt{x^2 + y^2}}\right)^2} = \sqrt{3}.$$

Thus,

$$\text{Area}(S) = 2\iint_{S_{xy}} \sqrt{3}\, dA = 2\sqrt{3}\int_{-\pi}^{\pi}\int_0^1 r\, dr\, d\theta = 2\sqrt{3}\int_{-\pi}^{\pi}\left\{\frac{r^2}{2}\right\}_0^1 d\theta = \sqrt{3}\,\{\theta\}_{-\pi}^{\pi} = 2\sqrt{3}\pi.$$

Alternatively,

$$\text{Area}(S) = 2\iint_{S_{xy}} \sqrt{3}\, dA = 2\sqrt{3}\,\text{Area}(S_{xy}) = 2\sqrt{3}(\pi).$$

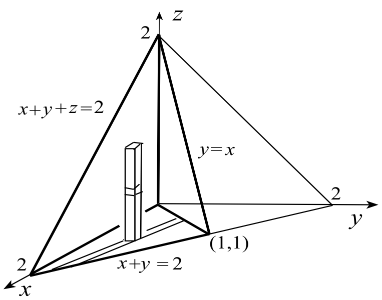
9. (a) Predict, with justification, whether the triple integral of the function $f(x, y, z) = z^4$ over the region V bounded by the surfaces

$$x + y + z = 2, \quad y = x, \quad y = 0, \quad z = 0$$

should be a positive number, a negative number, or zero.
(b) Evaluate the triple integral. **Hint**: Replace dV with $dz\,dx\,dy$.

(a) Since z^4 is non-negative throughout V, the integral should be a positive number.
(b) The value of the triple integral is

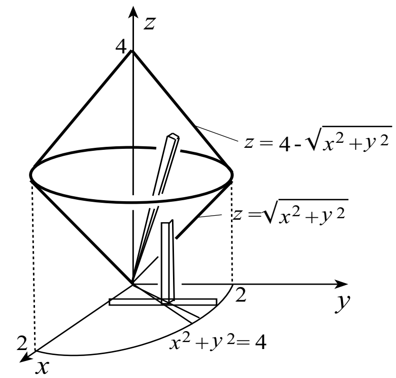
$$\begin{aligned}
\iiint_V z^4\,dV &= \int_0^1 \int_y^{2-y} \int_0^{2-x-y} z^4\,dz\,dx\,dy \\
&= \int_0^1 \int_y^{2-y} \frac{1}{5}(2-x-y)^5\,dx\,dy \\
&= \frac{1}{5}\int_0^1 \left\{-\frac{1}{6}(2-x-y)^6\right\}_y^{2-y} dy \\
&= \frac{1}{30}\int_0^1 (2-2y)^6\,dy = \frac{64}{30}\int_0^1 (1-y)^6\,dy \\
&= \frac{32}{15}\left\{-\frac{(1-y)^7}{7}\right\}_0^1 = \frac{32}{105}.
\end{aligned}$$



10. (a) **Set up but do not evaluate** triple iterated integrals in Cartesian, cylindrical, and spherical coordinates for the triple integral of the function $f(x, y, z) = x^2 + y^2$ over the volume V bounded by

$$z = \sqrt{x^2 + y^2}, \qquad z = 4 - \sqrt{x^2 + y^2}.$$

(b) Pick and evaluate (**only one**) of the triple iterated integrals in part (a).



(a) The cones intersect when

$$\sqrt{x^2 + y^2} = 4 - \sqrt{x^2 + y^2} \quad \Longrightarrow \quad \sqrt{x^2 + y^2} = 2,$$

or, $x^2 + y^2 = 4$. In Cartesian coordinates,

$$\iiint_V (x^2 + y^2)\, dV = \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{\sqrt{x^2+y^2}}^{4-\sqrt{x^2+y^2}} (x^2 + y^2)\, dz\, dy\, dx.$$

or,

$$4 \int_0^2 \int_0^{\sqrt{4-x^2}} \int_{\sqrt{x^2+y^2}}^{4-\sqrt{x^2+y^2}} (x^2 + y^2)\, dz\, dy\, dx.$$

In cylindrical coordinates,

$$\iiint_V (x^2 + y^2)\, dV = \int_{-\pi}^{\pi} \int_0^2 \int_r^{4-r} r^2\, r\, dz\, dr\, d\theta = \int_{-\pi}^{\pi} \int_0^2 \int_r^{4-r} r^3\, dz\, dr\, d\theta,$$

or,

$$4 \int_0^{\pi/2} \int_0^2 \int_r^{4-r} r^3\, dz\, dr\, d\theta.$$

In spherical coordinates,

$$\begin{aligned} \iiint_V (x^2 + y^2)\, dV &= \int_{-\pi}^{\pi} \int_0^{\pi/4} \int_0^{4/(\cos\phi + \sin\phi)} R^2 \sin^2\phi\, R^2 \sin\phi\, dR\, d\phi\, d\theta \\ &= \int_{-\pi}^{\pi} \int_0^{\pi/4} \int_0^{4/(\cos\phi + \sin\phi)} R^4 \sin^3\phi\, dR\, d\phi\, d\theta, \end{aligned}$$

or,

$$4 \int_0^{\pi/2} \int_0^{\pi/4} \int_0^{4/(\cos\phi + \sin\phi)} R^4 \sin^3\phi\, dR\, d\phi\, d\theta.$$

(b) If we choose to integrate with cylindrical coordinates,

$$\begin{aligned} \iiint_V (x^2 + y^2)\, dV &= \int_{-\pi}^{\pi} \int_0^2 \left\{r^3 z\right\}_r^{4-r} dr\, d\theta = \int_{-\pi}^{\pi} \int_0^2 [r^3(4 - r) - r^4]\, dr\, d\theta \\ &= \int_{-\pi}^{\pi} \int_0^2 (4r^3 - 2r^4)\, dr\, d\theta = \int_{-\pi}^{\pi} \left\{ r^4 - \frac{2r^5}{5} \right\}_0^2 d\theta = \frac{16}{5} \left\{\theta\right\}_{-\pi}^{\pi} = \frac{32\pi}{5}. \end{aligned}$$