Solutions to fall 2024 Exam

1. Show that the lines

$$L_1: \quad \begin{array}{c} x + 2y - 3z = 35, \\ x - y + 4z = -22, \end{array} \qquad\qquad L_2: \quad \begin{array}{l} x = 1 + 2t, \\ y = 2 + t, \\ z = -3 - t, \end{array}$$

intersect, and find the equation of the plane containing both lines.

When we substitute parametric equations for L_2 into the plane $x + 2y - 3z = 35$, we get

$$(1 + 2t) + 2(2 + t) - 3(-3 - t) = 35 \quad \Longrightarrow \quad t = 3.$$

This gives the point $(7, 5, -6)$. Since this point also satisfies $x - y + 4z = -22$, this is the point of intersection of the lines. A vector along L_1 is

$$\begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & 2 & -3 \\ 1 & -1 & 4 \end{vmatrix} = (5, -7, -3).$$

A vector normal to the required plane is

$$\begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 5 & -7 & -3 \\ 2 & 1 & -1 \end{vmatrix} = (10, -1, 19).$$

The equation of the required plane is

$$10(x - 7) - (y - 5) + 19(z + 6) = 0.$$

2. Set up, **but do NOT evaluate**, a definite integral for the length of the curve

$$z = 2x - y, \quad 2x^2 + 3y + z = 2$$

between the points $(2, -5, 9)$ and $(0, 1, -1)$. You need not simplify the integrand.

If we set $x = t$, then

$$z = 2t - y, \quad 2t^2 + 3y + z = 2.$$

Parametric equations for the curve are therefore

$$x = t, \quad y = 1 - t - t^2, \quad z = t^2 + 3t - 1, \quad 0 \le t \le 2.$$

The length of the curve is

$$\int_0^2 \sqrt{1 + (-1 - 2t)^2 + (2t + 3)^2}\, dt.$$

3. Find the rate of change of the function $f(x, y,) = xy + yz + xz$ at the point $(2, 1, 1)$ in the downward direction normal to the surface $x^2y + xyz = 6$.

$$\nabla f_{|(2,1,1)} = (y + z, x + z, x + y)_{|(2,1,1)} = (2, 3, 3)$$

A normal to the surface at $(2, 1, 1)$ is

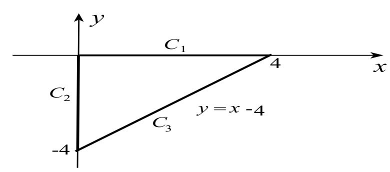
$$\nabla(x^2y + xyz - 6)_{|(2,1,1)} = (2xy + yz, x^2 + xz, xy)_{|(2,1,1)} = (5, 6, 2).$$

A unit vector to the surface in the downward direction is $\hat{\mathbf{n}} = \dfrac{-(5, 6, 2)}{\sqrt{65}}$. The required rate of change is

$$D_{\hat{\mathbf{n}}} f = \nabla f \cdot \left[\frac{-(5, 6, 2)}{\sqrt{65}}\right] = -\frac{34}{\sqrt{65}}.$$

4. Find the maximum value of the function $f(x, y) = 3xy^2 + 6xy + 3y$ on the region bounded by the curves

$$y = x - 4, \quad y = 0, \quad x = 0.$$



For critical points inside the triangle, we solve
$0 = f_x = 3y^2 + 6y = 3y(y + 2)$,
$0 = f_y = 6xy + 6x + 3 = 3(2xy + 2x + 1)$.
The only solutions are $(-1/2, 0)$, which is rejected, and $(1/2, -2)$ at which $f(1/2, -2) = \boxed{-6}$.
On C_1, $y = 0$ so that $f(x, 0) = \boxed{0}$. On C_2, $x = 0$ so that $f(0, y) = 3y, \quad -4 \le y \le 0$. Since there are no critical values, we evaluate $f(0, -4) = \boxed{-12}$ and $f(0, 0) = \boxed{0}$. On C_3, we set

$$g(y) = f(y + 4, y) = 3(y + 4)y^2 + 6y(y + 4) + 3y = 3y^3 + 18y^2 + 27y, \quad -4 \le y \le 0.$$

For critical values, we set

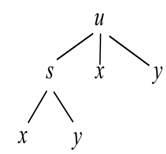
$$0 = g'(y) = 9y^2 + 36y + 27 = 9(y + 3)(y + 1).$$

Thus, $y = -1$ and $y = -3$, and we calculate $g(-1) = \boxed{-12}$ and $g(-3) = \boxed{0}$. The maximum value is 0.

5. If $f(x)$ is a differentiable function, show that the function $u(x, y) = f(4x - 3y) + 5(y - x)$ satisfies the equation

$$3\frac{\partial u}{\partial x} + 4\frac{\partial u}{\partial y} = 5.$$

If we set $s = 4x - 3y$, then
$u = f(s) + 5(y - x)$, and



$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial s}\frac{\partial s}{\partial x} + \frac{\partial u}{\partial x} = f'(s)(4) + (-5),$$
$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial s}\frac{\partial s}{\partial y} + \frac{\partial u}{\partial y} = f'(s)(-3) + (5).$$

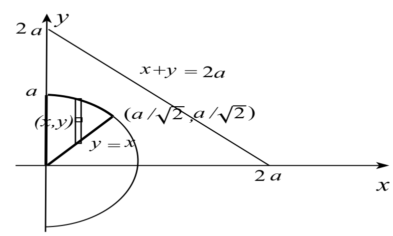
Thus

$$3\frac{\partial u}{\partial x} + 4\frac{\partial u}{\partial y} = 3[4f'(s) - 5] + 4[-3f'(s) + 5] = 5.$$

6. Set up, **but do NOT evaluate**, a double iterated integral to find the volume of the solid of revolution when the area bounded by the curves

$$x = \sqrt{a^2 - y^2}, \quad y = x, \quad x = 0$$

is rotated around the line $x + y = 2a$. ($a > 0$ is a constant.) Simplify the integrand as much as possible.



$$V = \int_0^{a/\sqrt{2}} \int_x^{\sqrt{a^2-x^2}} 2\pi \frac{|x + y - 2a|}{\sqrt{2}} \, dy \, dx$$
$$= \sqrt{2}\pi \int_0^{a/\sqrt{2}} \int_x^{\sqrt{a^2-x^2}} (2a - x - y) \, dy \, dx$$

In polar coordinates,

$$V = \int_{\pi/4}^{\pi/2} \int_0^a \frac{2\pi|r\cos\theta + r\sin\theta - 2a|}{\sqrt{2}} r \, dr \, d\theta = \sqrt{2}\pi \int_{\pi/4}^{\pi/2} \int_0^a (2a - r\cos\theta - r\sin\theta) \, r \, dr \, d\theta.$$

There is a second possible area, but it is more complicated.

7. Evaluate the double iterated integral

$$\int_{-1}^{0}\int_{-2x}^{2}\frac{x}{\sqrt{x^2+y^2}}dy\,dx.$$

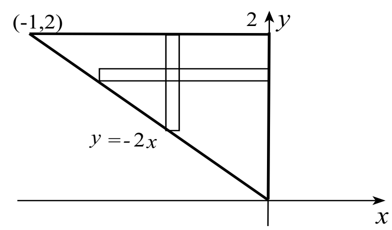
When we interchange the order of integration, and denote the integral by I,

$$I = \int_0^2\int_{-y/2}^{0}\frac{x}{x^2+y^2}\,dx\,dy$$

$$= \int_0^2 \left\{\sqrt{x^2+y^2}\right\}_{-y/2}^{0} dy$$

$$= \int_0^2 \left(y - \frac{\sqrt{5}y}{2}\right) dy$$

$$= \left(1-\frac{\sqrt{5}}{2}\right)\left\{\frac{y^2}{2}\right\}_0^2 = 2-\sqrt{5}.$$

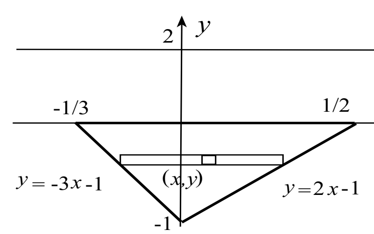


8. Set up, **but do NOT evaluate**, a double iterated integral to find the force due to water pressure on one side of the flat, vertical plate bounded by the curves

$$y = 2x - 1, \quad y = 0, \quad y + 3x = -1$$

where x and y are in metres. The surface of the water is 2 metres above the horizontal edge of the plate. Identify any physical constants in your solution.

$$F = \int_{-1}^{0}\int_{-(y+1)/3}^{(y+1)/2} 1000)(9.81)(2-y)\,dx\,dy \text{ N}$$



9. Consider the triple integral of a function $f(x,y,z)$ over the volume bounded by the surfaces

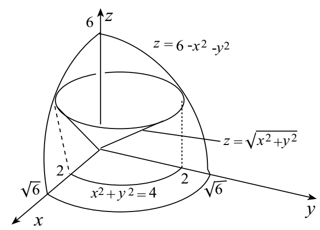
$$z = 6 - x^2 - y^2, \quad z = \sqrt{x^2 + y^2}.$$

(a) Set up, but do **NOT** evaluate, a triple iterated integral in Cartesian coordinates for the triple integral.

(b) Set up, but do **NOT** evaluate, a triple iterated integral in cylindrical coordinates for the triple integral.

(c) Set up, but do **NOT** evaluate, a triple iterated integral in spherical coordinates for the triple integral.

(d) Evaluate **ONE** of the integrals in (a), (b), and (c) if $f(x,y,z) = x^2 + y^2$. The choice of which one to evaluate is yours.



(a) $\displaystyle\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{\sqrt{x^2+y^2}}^{6-x^2-y^2} f(x,y,z)\,dz\,dy\,dx$

(b) $\displaystyle\int_{-\pi}^{\pi}\int_{0}^{2}\int_{r}^{6-r^2} f(r\cos\theta, r\sin\theta, z)\,r\,dz\,dr\,d\theta$

(c) $\displaystyle\int_{-\pi}^{\pi}\int_{0}^{\pi/4}\int_{0}^{6-R^2\sin^2\phi} f(R\cos\theta\sin\phi, R\sin\theta\sin\phi, R\cos\theta)R^2\sin\phi\,dR\,d\phi\,d\theta$

(d) Using part (b) when $f(x,y,z) = x^2 + y^2 = r^2$,

$$\int_{-\pi}^{\pi}\int_{0}^{2}\int_{r}^{6-r^2} r^3\,dz\,dr\,d\theta = \int_{-\pi}^{\pi}\int_{0}^{2}[r^2(6-r^2) - r^4]\,dr\,d\theta$$

$$= \int_{-\pi}^{\pi}\left\{\frac{6r^4}{4} - \frac{r^6}{6} - \frac{r^5}{5}\right\}_0^2 d\theta = \frac{104}{15}\{\theta\}_{-\pi}^{\pi} = \frac{208\pi}{15}.$$