Solutions to Final Exam Summer 2021

7 **1.** Set up, **but do NOT evaluate**, a definite integral for the length of the curve

$$y = z^2 + x, \quad x + y = 4$$

between the points $(0, 4, -2)$ and $(-1, 5, -\sqrt{6})$.

If we set $z = t$ in order to avoid square roots in the parametric equations, we obtain

$$x = 2 - \frac{t^2}{2}, \quad y = 2 + \frac{t^2}{2}, \quad z = t.$$

The length of the curve is given by the definite integral

$$\int_{-\sqrt{6}}^{-2} \sqrt{(-t)^2 + (t)^2 + 1^2}\, dt.$$

Alternatively, if we do set $x = t$, then parametric equations for the curve are

$$x = t, \quad y = 4 - t, \quad z = -\sqrt{4 - 2t}.$$

Length of the curve is

$$\int_{-1}^{0} \sqrt{1^2 + (-1)^2 + \left[\frac{1}{\sqrt{4 - 2t}}\right]^2}\, dt.$$

8 **2.** The line

$$2x + y = 4, \qquad x + y + z = 1$$

intersects the plane

$$3x + 4y - 2z = 5$$

at some point. Find the acute angle between a vector along the line and a vector perpendicular to the plane.

A vector along the line is $\mathbf{T} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 2 & 1 & 0 \\ 1 & 1 & 1 \end{vmatrix} = (1, -2, 1)$. A vector normal to the plane is $\mathbf{N} = (3, 4, -2)$. When we take scalar products of these vectors

$$\mathbf{T} \cdot \mathbf{N} = |\mathbf{T}||\mathbf{N}| \cos\theta,$$

where θ is an angle between the vectors. Hence,

$$\cos\theta = \frac{\mathbf{T} \cdot \mathbf{N}}{|\mathbf{T}||\mathbf{N}|} = \frac{3 - 8 - 2}{\sqrt{6}\sqrt{29}} = -\frac{7}{\sqrt{174}}.$$

Since this gives an obtuse angle, the acute angle is $\pi - \mathrm{Cos}^{-1}\left(\frac{-7}{\sqrt{174}}\right)$.

9 **3.** Find the point at which the normal line to the surface

$$x^2y^2 + xyz + 3x = 5,$$

at the point $(1, 2, -1)$ on the surface, intersects the plane

$$3x - 2y + z = 10.$$

A vector normal to the surface is

$$\nabla(x^2y^2 + xyz + 3x - 5)_{(1,2,-1)} = (2xy^2 + yz + 3, 2x^2y + xz, xy)_{(1,2,-1)} = (9, 3, 2).$$

Parametric equations for the normal line are

$$x = 1 + 9t, \quad y = 2 + 3t, \quad z = -1 + 2t.$$

When we substitute these into the equation of the plane,

$$10 = 3(1 + 9t) - 2(2 + 3t) + (-1 + 2t) = 23t - 2 \qquad \Longrightarrow \qquad t = \frac{12}{23}.$$

This gives the point $\left(\frac{131}{23}, \frac{82}{23}, \frac{1}{23}\right)$.

12 **4.** Find all critical points for the function

$$f(x, y) = y^4 + 3x^2y + x^2,$$

and classify them as giving relative maxima, relative minima, saddle points, or none of these.

For critical points, we solve

$$0 = f_x = 6xy + 2x = 2x(3y + 1), \quad 0 = f_y = 4y^3 + 3x^2.$$

They are $(0, 0)$ and $(\pm 2/9, -1/3)$. The second partial derivatives are

$$f_{xx} = 6y + 2, \qquad f_{xy} = 6x, \qquad f_{yy} = 12y^2.$$

At the critical points $(\pm 2/9, -1/3)$, $B^2 - AC = (4/3)^2 - 0(4/3) = 16/9 > 0$. Hence, these critical points yield saddle points. At $(0, 0)$, $B^2 - AC = 0$, so the test fails. If we write the function in the form $f(x, y) = y^4 + x^2(1 + 3y)$, we see that close to $(0, 0)$, the function is positive. Thus $(0, 0)$ gives a relative minimum.

10 **5.** The equations

$$u^2 - v = 3x + y, \qquad u - v^2 = x - 2y$$

define u and v as functions of x and y. Find $\partial u/\partial x$ at the values of x and y that give $u = 2$ and $v = 1$.

If we define $F(x, y, u, v) = u^2 - v - 3x - y$ and $G(x, y, u, v) = u - v^2 - x + 2y$, then

$$\frac{\partial u}{\partial x} = -\frac{\dfrac{\partial(F,G)}{\partial(x,v)}}{\dfrac{\partial(F,G)}{\partial(u,v)}} = -\frac{\begin{vmatrix} F_x & F_v \\ G_x & G_v \end{vmatrix}}{\begin{vmatrix} F_u & F_v \\ G_u & G_v \end{vmatrix}} = -\frac{\begin{vmatrix} -3 & -1 \\ -1 & -2v \end{vmatrix}}{\begin{vmatrix} 2u & -1 \\ 1 & -2v \end{vmatrix}}.$$

When $u = 2$ and $v = 1$,

$$\frac{\partial u}{\partial x} = -\frac{\begin{vmatrix} -3 & -1 \\ -1 & -2 \end{vmatrix}}{\begin{vmatrix} 4 & -1 \\ 1 & -2 \end{vmatrix}} = \frac{5}{7}.$$

4 **6.** Prove that the function

$$f(x, y, z) = xyz + x^2 y \sin\left(\frac{yz}{x^2}\right) - \frac{xy^3}{z}$$

satisfies the equation

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + z\frac{\partial f}{\partial z} = 3f(x, y, z).$$

Hint: It is not necessary to calculate the partial derivatives in order to prove the result. Think.

Since

$$f(tx, ty, tz) = (tx)(ty)(tz) + (tx)^2(ty)\sin\left(\frac{(ty)(tz)}{(tx)^2}\right) - \frac{(tx)(ty)^3}{tz} = t^3 f(x, y, z),$$

the function is homogeneous of degree 3. According to Euler's theorem,

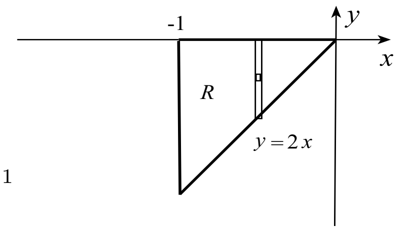
$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + z\frac{\partial f}{\partial z} = 3f(x, y, z).$$

12 **7.** Evaluate the double integral of the function $\dfrac{y}{\sqrt{x^2+y^2}}$ over the area bounded by the curves

$$y = 2x, \quad y = 0, \quad x = -1.$$

Using the figure to the right,

$$\begin{aligned}\iint_R \frac{y}{\sqrt{x^2+y^2}}dA &= \int_{-1}^{0}\int_{2x}^{0}\frac{y}{\sqrt{x^2+y^2}}dy\,dx \\ &= \int_{-1}^{0}\left\{\sqrt{x^2+y^2}\right\}_{2x}^{0}dx \\ &= \int_{-1}^{0}(-x+\sqrt{5}x)\,dx = (\sqrt{5}-1)\left\{\frac{x^2}{2}\right\}_{-1}^{0} \\ &= \frac{1}{2}(1-\sqrt{5}).\end{aligned}$$



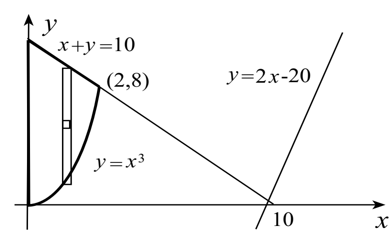
10 **8.** Set up, **but do NOT evaluate**, a double iterated integral to find the volume of the solid of revolution when the area bounded by the curve

$$y = x^3, \qquad x + y = 10, \qquad x = 0$$

is rotated around the line $2x - y = 20$. Simplify the integrand as much as possible.

The volume is

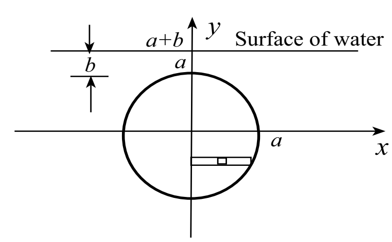
$$\begin{aligned}V &= \int_0^2\int_{x^3}^{10-x} 2\pi\frac{|2x-y-20|}{\sqrt{5}}dy\,dx \\ &= \frac{2\pi}{\sqrt{5}}\int_0^2\int_{x^3}^{10-x}(20-2x+y)\,dy\,dx.\end{aligned}$$



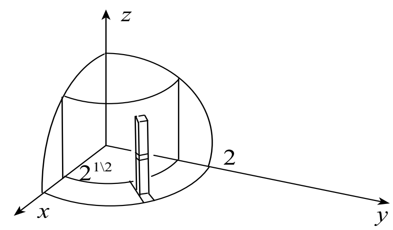
10 **9.** Set up, **but do NOT evaluate**, a double iterated integral to find the force due to water pressure on one side of a vertical, circular plate of radius a when the uppermost point of the plate is b units below the surface.

The force is

$$\begin{aligned}F &= 2\int_{-a}^{a}\int_0^{\sqrt{a^2-y^2}}\rho g(a+b-y)dx\,dy \\ &= 2(9.81)(1000)\int_{-a}^{a}\int_0^{\sqrt{a^2-y^2}}(a+b-y)\,dx\,dy.\end{aligned}$$



9 **10.** Set up, **but do NOT evaluate**, an iterated integral to find the volume inside the surface $x^2 + y^2 + z^2 = 4$, but outside the surface $x^2 + y^2 = 2$.

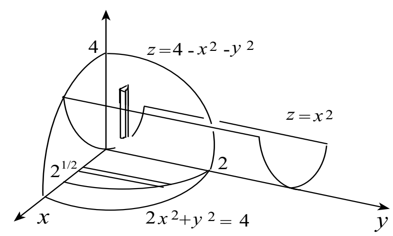


The volume is

$$V = 8 \int_0^{\pi/2} \int_{\sqrt{2}}^{2} \int_0^{\sqrt{4-r^2}} r\, dz\, dr\, d\theta.$$

9 **11.** Set up, **but do NOT evaluate**, an iterated integral to find the first moment about the xy-plane of a mass of constant density ρ bounded by the surfaces

$$z = x^2, \qquad z = 4 - x^2 - y^2.$$



The moment is

$$4 \int_0^{\sqrt{2}} \int_0^{\sqrt{4-2x^2}} \int_{x^2}^{4-x^2-y^2} \rho\, z\, dz\, dy\, dx.$$