MATH 2130 Test 1 October 10, 2012 60 minutes

Values

10 **1.** Show that the lines

$$x = 1 + t, \quad y = 2 + 3t, \quad z = 4 - 2t; \qquad \text{and} \qquad y = 3x, \quad 2x + z = 9$$

determine a plane, and find its equation simplified as much as possible.

A vector along the first line is $\mathbf{v}_1 = \langle 1, 3, -2 \rangle$. Since parametric equations for the second line are $x = u$, $y = 3u$, $z = 9 - 2u$, a vector along this line is $\mathbf{v_2} = \langle 1, 3, -2 \rangle$. The lines are therefore parallel, and since they are not the same line, they determine a plane. A point on each line is $P_1(1, 2, 4)$ and $P_2(0, 0, 9)$. Since $\mathbf{v}_1$ and $\mathbf{P_1P_2} = \langle -1, -2, 5 \rangle$ are vectors in the plane, a normal to the plane is

$$\mathbf{v}_1 \times \mathbf{P_1P_2} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & 3 & -2 \\ -1 & -2 & 5 \end{vmatrix} = \langle 11, -3, 1 \rangle.$$

The equation of the plane is

$$11(x - 0) - 3(y - 0) + 1(z - 9) = 0 \qquad \text{or} \qquad 11x - 3y + z = 9.$$

6 **2.** Find the distance between the line $x = 3 + 2t$, $y = -1 + t$, $z = 5 + 4t$ and the plane $6x - 8y - z = 7$.

If we substitute the parametric equations of the line into the equation of the plane, we obtain

$$6(3 + 2t) - 8(-1 + t) - (5 + 4t) = 7 \qquad \Longrightarrow \qquad 21 = 7.$$

This shows that the line is parallel to the plane. Since $(3, -1, 5)$ is a point on the line, the distance is

$$\frac{|6(3) - 8(-1) - (5) - 7|}{\sqrt{6^2 + (-8)^2 + (-1)^2}} = \frac{14}{\sqrt{101}}.$$

5 **3.** Find a vector of length 3 tangent to the curve

$$x = t^3 + t, \quad y = 2t - t^2, \quad z = t + 1,$$

at the point $(2, 1, 2)$.

A tangent vector to the curve is

$$\mathbf{T} = (3t^2 + 1)\hat{\mathbf{i}} + (2 - 2t)\hat{\mathbf{j}} + \hat{\mathbf{k}}.$$

Since $t = 1$ gives the point $(2, 1, 2)$ on the curve, a tangent vector at this point is

$$\mathbf{T}(1) = 4\hat{\mathbf{i}} + \hat{\mathbf{k}}.$$

A unit tangent vector at the point is

$$\hat{\mathbf{T}}(1) = \frac{4\hat{\mathbf{i}} + \hat{\mathbf{k}}}{\sqrt{17}}.$$

A tangent vector of length 3 is

$$\frac{3}{\sqrt{17}}(4\hat{\mathbf{i}} + \hat{\mathbf{k}}).$$

8 **4.** Find parametric equations for the curve

$$z = 4x^2 + y^2, \quad 8x + 4y + z = 8,$$

directed clockwise as viewed from a point far up the z-axis.

If we solve the second equation for z and equate results,

$$\begin{aligned} 4x^2 + y^2 &= 8 - 8x - 4y \\ 4x^2 + 8x + y^2 + 4y &= 8 \\ 4(x+1)^2 + (y+2)^2 &= 16 \\ \frac{(x+1)^2}{4} + \frac{(y+2)^2}{16} &= 1. \end{aligned}$$

We therefore set

$$\begin{aligned} x &= -1 + 2\cos t, \\ y &= -2 + 4\sin t, \\ z &= 8 - 8(-1 + 2\cos t) - 4(-2 + 4\sin t) = 24 - 16\cos t - 16\sin t, \qquad 0 \le t \le 2\pi. \end{aligned}$$

Since these give the curve in the opposite direction to that required, we replace t by $-t$,

$$x = -1 + 2\cos t, \quad y = -2 - 4\sin t, \quad z = 24 - 16\cos t + 16\sin t, \quad 0 \le t \le 2\pi.$$

6 **5.** Show that the following limit does not exist,

$$\lim_{(x,y)\to(0,0)} \frac{xy^3}{x^2+y^6}.$$

If we approach $(0,0)$ along the cubic curves $x = ay^3$, then

$$\lim_{(x,y)\to(0,0)} \frac{xy^3}{x^2+y^6} = \lim_{y\to 0} \frac{ay^6}{a^2y^6+y^6} = \frac{a}{a^2+1}.$$

Since this result depends on a, it follows that the original limit does not exist.

5 **6.** Show that the function $f(x,y) = x^2 + y^2 e^{y/x}$ satisfies the equation

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = 2f(x,y).$$

$$\begin{aligned} x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} &= x\left[2x + y^2e^{y/x}\left(-\frac{y}{x^2}\right)\right] + y\left[2ye^{y/x} + y^2e^{y/x}\left(\frac{1}{x}\right)\right] \\ &= 2(x^2 + y^2e^{y/x}) \\ &= 2f(x,y). \end{aligned}$$