MATH 2130 Test 1 October 13, 2020 60 minutes

8 **1.** Show that the lines

$$L_1: \quad \begin{aligned} x &= 3+t, \\ y &= 2-t, \\ z &= 4+2t; \end{aligned} \qquad \text{and} \qquad L_2: \quad \begin{aligned} x+2y-3z &= 4, \\ 2x-y &= 6 \end{aligned}$$

are not parallel, and find the shortest distance between them.

A vector along L_1 is $\mathbf{v}_1 = (1,-1,2)$. A vector along L_2 is

$$\mathbf{v}_2 = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & 2 & -3 \\ 2 & -1 & 0 \end{vmatrix} = (-3,-6,-5).$$

Since these vectors are not multiples of each other, the lines are not parallel.

A point on L_1 is $R(3,2,4)$, and a point on L_2 is $S(0,-6,-16/3)$. A vector normal to both lines is

$$\mathbf{N} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & -1 & 2 \\ 3 & 6 & 5 \end{vmatrix} = (-17,1,9).$$

The distance between the lines is

$$\left|\mathbf{RS}\cdot\hat{\mathbf{N}}\right| = \left|(-3,-8,-28/3)\cdot\frac{(-17,1,9)}{\sqrt{17^2+1+81}}\right| = \frac{41}{\sqrt{371}}.$$

7 **2.** (a) Find parametric equations for the curve

$$z = 2x^2+y^2, \quad x^2+(y-1)^2 = 4.$$

Direction along the curve is your choice.

(b) Use your equations in part (a) to set up a definite integral for the length of the curve. Do **NOT** evaluate the integral, nor simplify the integrand.

(a) Parametric equations for the curve are

$$x = 2\cos t, \quad y = 1+2\sin t, \quad z = 8\cos^2 t + (1+2\sin t)^2, \qquad 0 \le t \le 2\pi.$$

(b) The length of the curve is given by

$$\int_0^{2\pi} \sqrt{(-2\sin t)^2 + (2\cos t)^2 + [-16\cos t\sin t + 2(1+2\sin t)(2\cos t)]^2}\, dt.$$

5 **3.** Evaluate the following limit, or show that it does not exist.

$$\lim_{(x,y)\to(0,0)} \frac{x^2 - 3xye^x + y^2}{x^2 + y^2}$$

If we approach the origin along straight lines by setting $y = mx$, then along these lines

$$\lim_{(x,y)\to(0,0)} \frac{x^2 - 3xye^x + y^2}{x^2 + y^2} = \lim_{x\to 0}\left(\frac{x^2 - 3mx^2e^x + m^2x^2}{x^2 + m^2x^2}\right) = \lim_{x\to 0}\left(\frac{1 - 3me^x + m^2}{1 + m^2}\right)$$
$$= \frac{1 - 3m + m^2}{1 + m^2}.$$

Since this result depends on m, the original limit does not exist.

6 **4.** Find the components of the gradient of the function

$$f(x, y, z) = x^2y + 3xz - 4$$

at the point $(2, -1, 3)$ in directions normal to the surface

$$x^2yz + 3xy + z + 15 = 0.$$

$$\nabla f_{|(2,-1,3)} = (2xy + 3z, x^2, 3x)_{|(2,-1,3)} = (5, 4, 6)$$

A normal vector to the surface is

$$\nabla(x^2yz + 3xy + z + 15)_{|(2,-1,3)} = (2xyz + 3y, x^2z + 3x, x^2y + 1)_{|(2,-1,3)} = (-15, 18, -3).$$

So also is $(5, -6, 1)$. The components of the gradient in the directions $\pm(5, -6, 1)$ are

$$(5, 4, 6) \cdot \left[\frac{\pm(5, -6, 1)}{\sqrt{62}}\right] = \frac{\pm 7}{\sqrt{62}}.$$

4 **5.** Find the unit tangent vector to the curve

$$x = (t-1)^3, \quad y = 3(t-1)^2, \quad z = -(t-1)^2$$

at the point $(0,0,0)$.

A tangent vector to the curve is

$$\mathbf{T} = \big(3(t-1)^2, 6(t-1), -2(t-1)\big) = (t-1)\big(3(t-1), 6, -2\big).$$

If we remove the $t-1$ before the vector, we change its length, but not its direction. Thus, a tangent vector to the curve is also $\mathbf{T}_1 = (3t-3, 6, -2)$. Since $t = 1$ gives the point $(0,0,0)$ on the curve, a tangent vector at the origin is $(0,6,-2)$, and the unit tangent vector is $(0,6,-2)/\sqrt{40} = (0,3,-1)/\sqrt{10}$.

4 **6.** Prove that when $f(t)$ is a differentiable scalar function, and $\mathbf{v}(t)$ is a vector function with differentiable components, then

$$\frac{d}{dt}[f(t)\mathbf{v}(t)] = \frac{df}{dt}\mathbf{v} + f(t)\frac{d\mathbf{v}}{dt}.$$

If we set $\mathbf{v} = (v_1, v_2, v_3)$, then

$$\frac{d}{dt}[f(t)\mathbf{v}(t)] = \frac{d}{dt}\big(f(t)v_1\hat{\mathbf{i}} + f(t)v_2\hat{\mathbf{j}} + f(t)v_3\hat{\mathbf{k}}\big).$$

To differentiate a vector, we differentiate its components,

$$\begin{aligned}\frac{d}{dt}[f(t)\mathbf{v}(t)] &= \left(\frac{df}{dt}v_1 + f(t)\frac{dv_1}{dt}\right)\hat{\mathbf{i}} + \left(\frac{df}{dt}v_2 + f(t)\frac{dv_2}{dt}\right)\hat{\mathbf{j}} + \left(\frac{df}{dt}v_3 + f(t)\frac{dv_3}{dt}\right)\hat{\mathbf{k}}\\ &= \frac{df}{dt}(v_1\hat{\mathbf{i}} + v_2\hat{\mathbf{j}} + v_3\hat{\mathbf{k}}) + f(t)\left(\frac{dv_1}{dt}\hat{\mathbf{i}} + \frac{dv_2}{dt}\hat{\mathbf{j}} + \frac{dv_3}{dt}\hat{\mathbf{k}}\right)\\ &= \frac{df}{dt}\mathbf{v} + f(t)\frac{d\mathbf{v}}{dt}.\end{aligned}$$