MATH 2130 Test 1 October 5, 2021 60 minutes

4 **1.** Find the projection of the curve

$$x + y + z = 1, \quad x^2 - y^2 - z = 0$$

in the xy-plane. Is the projection a circle, ellipse, parabola, hyperbola, or none of these?

When we eliminate z between the equations,

$$x + y + x^2 - y^2 = 1 \qquad \Longrightarrow \qquad (x + 1/2)^2 - (y - 1/2)^2 = 1.$$

This is the equation of a hyperbola.

14 **2.** (a) Prove that the following lines are not parallel, and find the shortest distance between them.

$$L_1: \quad \begin{aligned} x &= 1 - 2t, \\ y &= 3 + 5t, \\ z &= -2 + t; \end{aligned} \qquad \text{and} \qquad L_2: \quad \frac{x-4}{-2} = \frac{2y+6}{5} = z + 5.$$

(b) Explain why it is unnecessary to prove in part (a) that the lines do not intersect.

(a) A vector along L_1 is $\mathbf{v}_1 = (-2, 5, 1)$, and a vector along L_2 is $(-2, 5/2, 1)$ or $\mathbf{v}_2 = (-4, 5, 2)$. Because neither vector is a multiple of the other, the vectors are not parallel, and therefore the lines are not parallel.

A vector perpendicular to both lines is

$$\mathbf{N} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ -2 & 5 & 1 \\ -4 & 5 & 2 \end{vmatrix} = (5, 0, 10).$$

Since $P(1, 3, -2)$ is a point on L_1, and $Q(4, -3, -5)$ is a point on L_2, the shortest distance between the lines is

$$\left|\mathbf{PQ} \cdot \hat{\mathbf{N}}\right| = \left|(3, -6, -3) \cdot \frac{(1, 0, 2)}{\sqrt{5}}\right| = \frac{3}{\sqrt{5}}.$$

(b) If the lines were to intersect, the distance in part (a) would turn out to be 0.

10 **3.** Find all unit tangent vectors to the curve

$$z = 3 - 2x^2 - 2y^2, \quad x + y^2 = 4$$

at the point $(3, -1, -17)$.

Parametric equations for the curve are

$$x = 4 - t^2, \quad y = t, \quad z = 3 - 2(4 - t^2)^2 - 2t^2 = -29 + 14t^2 - 2t^4.$$

A tangent vector to the curve at any point is

$$\mathbf{T}(t) = (-2t, 1, 28t - 8t^3).$$

Since $t = -1$ gives the point $(3, -1, -17)$ on the curve, a tangent vector at this point is

$$\mathbf{T}(-1) = (2, 1, -20).$$

The two unit tangent vectors to the curve at $(3, -1, -17)$ are

$$\pm\frac{(2, 1, -20)}{\sqrt{405}}.$$

12 **4.** (a) Prove that the following lines intersect

$$L_1: \quad \begin{aligned} x + 2y + z &= -1, \\ x + y &= 1; \end{aligned} \qquad\qquad L_2: \quad \begin{aligned} x &= 1 + 2t, \\ y &= 3 - t, \\ z &= 1 + 3t. \end{aligned}$$

(b) Find the equation of the plane that contains the lines. Simplify the equation.

(a) If we substitute $x = 1 + 2t$ and $y = 3 - t$ into $x + y = 1$, we get

$$(1 + 2t) + (3 - t) = 1 \qquad \Longrightarrow \qquad t = -3.$$

Substitution of $t = -3$ into the parametric equations for L_2 gives the point $(-5, 6, -8)$. Since this point satisfies the equations for L_2, it is the point of intersection of the lines.

(b) A vector along L_1 is

$$\mathbf{v}_1 = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & 2 & 1 \\ 1 & 1 & 0 \end{vmatrix} = (-1, 1, -1).$$

Since a vector along L_2 is $\mathbf{v}_2 = (2, -1, 3)$, a vector normal to the required plane is

$$\begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ -1 & 1 & -1 \\ 2 & -1 & 3 \end{vmatrix} = (2, 1, -1).$$

The equation of the plane is therefore

$$2(x + 5) + (y - 6) - (z + 8) = 0 \qquad \Longrightarrow \qquad 2x + y - z = 4.$$