MATH 2130 Test 1 October 10, 2023 70 minutes

6 **1.** Find equations for the projection of the curve

$$z = x^2 + y^2, \quad 3x + 2y + z = 4$$

in the xy-plane. Describe the projection in detail.

Equations for the projection are

$$x^2 + y^2 = 4 - 3x - 2y, \qquad z = 0.$$

When we complete squares in the first equation,

$$(x + 3/2)^2 + (y + 1)^2 = 4 + 9/4 + 1 = 29/4.$$

This is the equation of a circle with centre $(-3/2, -1)$ and radius $\sqrt{29}/2$.

6 **2.** Set up, but do **NOT** evaluate, a definite integral for the length of the curve

$$x + y + z = 4, \qquad x^2 + y^2 = 4y$$

You need not simplify the integrand.

If we set $x = 2\cos t$ and $y = 2 + 2\sin t$, then

$$z = 4 - 2\cos t - 2 - 2\sin t = 2 - 2\cos t - 2\sin t, \quad 0 \le t \le 2\pi.$$

The length of the curve is given by

$$\begin{aligned} L &= \int_0^{2\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt \\ &= \int_0^{2\pi} \sqrt{(-2\sin t)^2 + (2\cos t)^2 + (2\sin t - 2\cos t)^2}\, dt. \end{aligned}$$

12 **3.** Find, if possible, the equation for the plane that contains the lines

$$L_1: \quad \begin{aligned} x &= -1 + 2t, \\ y &= -5 + 3t, \\ z &= 2 - t; \end{aligned} \qquad \text{and} \qquad L_2: \quad \begin{aligned} x + y - z &= -2, \\ 2x - 3y + 2z &= 10. \end{aligned}$$

A vector along L_1 is $\mathbf{v}_1 = (2, 3, -1)$. A vector along L_2 is

$$\mathbf{v}_2 = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & 1 & -1 \\ 2 & -3 & 2 \end{vmatrix} = (-1, -4, -5).$$

Since these vectors are not multiples, the lines are not parallel. To see whether they intersect, we set

$$(-1 + 2t) + (-5 + 3t) - (2 - t) = -2 \quad \Longrightarrow \quad 6t = 6 \quad \Longrightarrow \quad t = 1.$$

This gives the point $(1, -2, 1)$ as the point of intersection of L_1 and the plane $x + y - z = 2$. Since this point also satisfies $2x - 3y + 2z = 10$, the lines intersect at $(1, -2, 1)$. A normal vector to the required plane is

$$\begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 2 & 3 & -1 \\ 1 & 4 & 5 \end{vmatrix} = (19, -11, 5).$$

The equation of the required plane is

$$19(x - 1) - 11(y + 2) + 5(z - 1) = 0.$$

10 **4.** Find the distance between the lines

$$L_1: \quad \begin{array}{l} x = 1 + 2t, \\ y = -2 + t, \\ z = 3 - 4t; \end{array} \qquad \text{and} \qquad L_2: \quad \begin{array}{l} 2x + y - z = -3, \\ x - y + 2z = 9. \end{array}$$

Vectors along these lines are

$$\mathbf{v}_1 = (2, 1, -4) \qquad \text{and} \qquad \mathbf{v}_2 = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 2 & 1 & -1 \\ 1 & -1 & 2 \end{vmatrix} = (1, -5, -3).$$

Since these vectors are not multiples, the lines are not parallel. Points on these lines are $R(1, -2, 3)$ and $S(2, -7, 0)$. The distance between the lines is $d = |\mathbf{RS} \cdot \widehat{\mathbf{PQ}}|$, where $\mathbf{PQ}$ is a vector normal to both lines, one such being

$$\begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 2 & 1 & -4 \\ 1 & -5 & -3 \end{vmatrix} = (-23, 2, -11).$$

Thus,

$$d = \left| (1, -5, -3) \cdot \frac{(-23, 2, -11)}{\sqrt{654}} \right| = 0.$$

The lines therefore intersect.

6 **5.** Find all unit tangent vectors to the curve

$$x + y^2 = 1, \qquad z + y^3 + 2x = 2$$

at the point $(0, 1, 1)$.

If we set $y = t$, then $x = 1 - t^2$, and $z = 2 - t^3 - 2(1 - t^2) = 2t^2 - t^3$. A tangent vector to the curve is $\mathbf{T} = (-2t, 1, 4t - 3t^2)$. At $0, 1, 1)$, $t = 1$, so that

$$\mathbf{T}(1) = (-2, 1, 1).$$

Unit tangent vectors are

$$\pm\hat{\mathbf{T}}(1) = \frac{\pm(-2, 1, 1)}{\sqrt{6}}.$$