Solutions to MATH 2130 Test 1 May 25, 2021

8 **1.** (a) Find all unit tangent vectors to the curve

$$x = 3t^2, \quad y = 2t^3, \qquad z = 4\sin(\pi t^2)$$

at the point $(0,0,0)$.

(b) Set up, but do **NOT** evaluate, a definite integral for the length of the curve between the points $(0,0,0)$ and $(3,-2,0)$.

(a) Tangent vectors to the curve are

$$\begin{aligned}\mathbf{T}(t) &= \frac{dx}{dt}\hat{\mathbf{i}} + \frac{dy}{dt}\hat{\mathbf{j}} + \frac{dz}{dt}\hat{\mathbf{k}}\\ &= 6t\hat{\mathbf{i}} + 6t^2\hat{\mathbf{j}} + 8\pi t\cos(\pi t^2)\hat{\mathbf{k}}\\ &= t[6\hat{\mathbf{i}} + 6t\hat{\mathbf{j}} + 8\pi\cos(\pi t^2)\hat{\mathbf{k}}].\end{aligned}$$

The multiplicative t can be removed without changing the direction of the vector. In other words, we can take tangent vectors as

$$\mathbf{T}_1(t) = 6\hat{\mathbf{i}} + 6t\hat{\mathbf{j}} + 8\pi\cos(\pi t^2)\hat{\mathbf{k}}.$$

Since $t = 0$ gives the point $(0,0,0)$, a tangent vector at $(0,0,0)$ is

$$\mathbf{T}_1(0) = 6\hat{\mathbf{i}} + 8\pi\hat{\mathbf{k}}.$$

The two unit tangent vectors to the curve at $(0,0,0)$ are

$$\pm\frac{1}{\sqrt{36+64\pi^2}}(6\hat{\mathbf{i}} + 8\pi\hat{\mathbf{k}}).$$

(b) Since $t = -1$ gives the point $(3,-2,0)$, the length of the curve is

$$\int_{-1}^{0}\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,dt = \int_{-1}^{0}\sqrt{(6t)^2 + (6t^2)^2 + [8\pi t\cos(\pi t^2)]^2}\,dt.$$

11 **2.** Find parametric equations for the projection of the curve

$$y^2 + 4z^2 = 16, \qquad x + y = 3$$

in the xz-plane directed so that x increases when z is positive.

The projection of the curve in the xz-plane has equation

$$(3 - x)^2 + 4z^2 = 16 \quad \text{or} \quad (x - 3)^2 + 4z^2 = 16,$$

an ellipse. Parametric equations for the ellipse are

$$x = 3 + 4\cos t, \quad z = 2\sin t, \quad 0 \le t \le 2\pi.$$

Since these equations trace the ellipse in the wrong direction, we replace t by $-t$,

$$x = 3 + 4\cos t, \quad y = -2\sin t, \qquad 0 \le -t \le 2\pi \quad \Longrightarrow \quad 0 \ge t \ge -2\pi \quad -2\pi \le t \le 0.$$

We can replace these values of t with $0 \le t \le 2\pi$.

6 **3.** Evaluate the following limit, or show that it does not exist

$$\lim_{(x,y)\to(0,0)} \frac{x^3 y}{y^2 + 2x^6}$$

If we approach $(0,0)$ along the cubic curves $y = ax^3$,

$$\lim_{(x,y)\to(0,0)} \frac{x^3 y}{y^2 + 2x^6} = \lim_{x\to 0} \frac{x^3(ax^3)}{(ax^3)^2 + 2x^6} = \lim_{x\to 0} \frac{a}{a^2 + 2} = \frac{a}{a^2 + 2}.$$

Since this limit depends on a, the value of the original limit depends on the mode of approach to $(0,0)$. The original limit does not therefore exist.

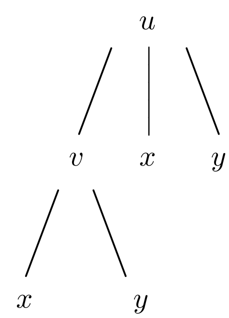
15 **4.** If $f(v)$ is a differentiable function, show that the function $u(x, y) = f(5x^2 + 6y^2) + 2x + 3y$ satisfies the equation

$$6y\frac{\partial u}{\partial x} - 5x\frac{\partial u}{\partial y} = 12y - 15x.$$

If we set $v = 5x^2 + 6y^2$, then

$$u = f(v) + 2x + 3y, \quad \text{where } v = 5x^2 + 6y^2.$$

From the schematic,



we find

$$\left.\frac{\partial u}{\partial x}\right)_y = \frac{\partial u}{\partial v}\frac{\partial v}{\partial x} + \left.\frac{\partial u}{\partial x}\right)_{v,y}$$
$$= f'(v)(10x) + 2,$$
$$\left.\frac{\partial u}{\partial y}\right)_x = \frac{\partial u}{\partial v}\frac{\partial v}{\partial y} + \left.\frac{\partial u}{\partial y}\right)_{v,x}$$
$$= f'(v)(12y) + 3.$$

Thus,

$$6y\frac{\partial u}{\partial x} - 5x\frac{\partial u}{\partial y} = 6y[f'(v)(10x) + 2] - 5x[f'(v)(12y) + 3] = 12y - 15x.$$