MATH 2130 Test 1 Winter 2012 **60 minutes**

10 **1.** Find the equation of the plane containing the lines

$$x = 2t, \quad y = 4 - 3t, \quad z = 3 + t; \qquad \text{and} \qquad x + 2y - 3z = -8, \quad 3x - y + z = 9.$$

Simplify the equation as much as possible.

A vector along the first line is $\mathbf{v}_1 = (2, -3, 1)$. A vector along the second line is

$$\mathbf{v}_2 = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & 2 & -3 \\ 3 & -1 & 1 \end{vmatrix} = (-1, -10, -7).$$

The lines are therefore not parallel. If we substitute the parametric equations of the first line into $x + 2y - 3z = -8$, we get

$$2t + 2(4 - 3t) - 3(3 + t) = -8 \quad \Longrightarrow \quad -7t = -7 \quad \Longrightarrow \quad t = 1.$$

The point $(2, 1, 4)$ satisfies equations for both lines and is therefore their point of intersection. A vector normal to the required plane is

$$\begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 2 & -3 & 1 \\ 1 & 10 & 7 \end{vmatrix} = (-31, -13, 23).$$

The equation of the plane is

$$31(x - 2) + 13(y - 1) - 23(z - 4) = 0 \quad \Longrightarrow \quad 31x + 13y - 23z = -17.$$

10 **2.** Find the distance between the lines $x = 2 + 4t$, $y = 1 + 7t$, $z = -3 + 5t$ and $2x + y - 3z = 8$, $x - 2y + 2z = -1$.

A vector along the first line is $\mathbf{v}_1 = (4, 7, 5)$. A vector along the second line is

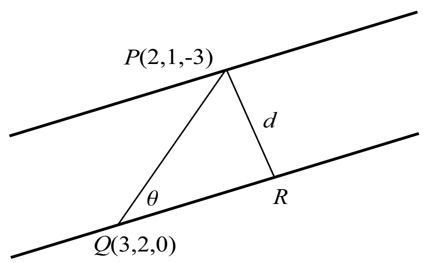
$$\mathbf{v}_2 = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 2 & 1 & -3 \\ 1 & -2 & 2 \end{vmatrix} = (-4, -7, -5).$$

The lines are parallel. Points on the lines are $P(2, 1, -3)$ and $Q(3, 2, 0)$. The required distance is

$$d = |\mathbf{PQ}| \sin\theta = |\mathbf{PQ}||\hat{\mathbf{QR}}| \sin\theta = |\mathbf{PQ} \times \hat{\mathbf{QR}}|.$$

Since $\mathbf{PQ} = (1, 1, 3)$ and $\hat{\mathbf{QR}} = (4, 7, 5)/(3\sqrt{10})$

$$d = \frac{1}{3\sqrt{10}} \left\| \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & 1 & 3 \\ 4 & 7 & 5 \end{vmatrix} \right\| = \frac{1}{3\sqrt{10}} |(-16, 7, 3)| = \frac{\sqrt{314}}{3\sqrt{10}}.$$



8 **3.** Find the length of the curve

$$x = t^2, \quad y = t^3 + 3, \quad z = t^2 - 2,$$

between the points $(1, 4, -1)$ and $(1, 2, -1)$.

Since values of t giving the points are $t = -1$ and $t = 1$,

$$L = \int_{-1}^{1} \sqrt{(2t)^2 + (3t^2)^2 + (2t)^2}\, dt = \int_{-1}^{1} \sqrt{8t^2 + 9t^4}\, dt$$

$$= 2\int_{0}^{1} t\sqrt{8 + 9t^2}\, dt = 2\left\{\frac{1}{27}(8 + 9t^2)^{3/2}\right\}_0^1 = \frac{2}{27}(17\sqrt{17} - 8\sqrt{8}).$$

8 **4.** Find a unit tangent vector to the curve

$$x^2 + y = 4, \quad z - 3x = 5,$$

at the point $(-2, 0, -1)$. Coordinate z must decrease along the curve.

Parametric equations for the curve are

$$x = -t, \quad y = 4 - t^2, \quad z = 5 - 3t.$$

A tangent vector at any point on the curve is

$$\mathbf{T} = -\hat{\mathbf{i}} - 2t\hat{\mathbf{j}} - 3\hat{\mathbf{k}}.$$

Since $t = 2$ gives the point $(-2, 0, -1)$,

$$\mathbf{T}(2) = -\hat{\mathbf{i}} - 4\hat{\mathbf{j}} - 3\hat{\mathbf{k}} \quad \Longrightarrow \quad \hat{\mathbf{T}}(2) = \frac{-\hat{\mathbf{i}} - 4\hat{\mathbf{j}} - 3\hat{\mathbf{k}}}{\sqrt{26}}.$$

4 **5.** Show that if $f(t) < 0$ for $-1 \le t \le 2$, then the curve

$$x = 2f(t)\cos t, \quad y = 3f(t)\sin t, \quad z = f(t),$$

lies on a cone. What is the equation of the cone?

$$\left(\frac{x}{2}\right)^2 + \left(\frac{y}{3}\right)^2 = [f(t)]^2\cos^2 t + [f(t)]^2\sin^2 t = [f(t)]^2 = z^2$$

Since $z < 0$, we must have $z = -\sqrt{\dfrac{x^2}{4} + \dfrac{y^2}{9}}$, an elliptic cone.