**MATH 2130 Test 2 2010 Solutions**

**1.** Find the rate of change of the function $f(x,y,z) = x^3y + y^2z + xyz + z^2$ in the direction of the curve

$$\mathbf{r}(t) = (\cos t + 1)\hat{\mathbf{i}} + \sin t\,\hat{\mathbf{j}} + (2\sin 2t + 2)\hat{\mathbf{k}}$$

at the point $(1,-1,2)$.

$$\nabla f_{|(1,-1,2)} = [(3x^2y + yz)\hat{\mathbf{i}} + (x^3 + 2yz + xz)\hat{\mathbf{j}} + (y^2 + xy + 2z)\hat{\mathbf{k}}]_{|(1,-1,2)} = -5\hat{\mathbf{i}} - \hat{\mathbf{j}} + 4\hat{\mathbf{k}}$$

A tangent vector to the curve is

$$\mathbf{T}(t) = (-\sin t)\hat{\mathbf{i}} + (\cos t)\hat{\mathbf{j}} + (4\cos 2t)\hat{\mathbf{k}}.$$

Since $t = 3\pi/2$ gives the point $(1,-1,2)$,

$$\mathbf{T}(3\pi/2) = \hat{\mathbf{i}} - 4\hat{\mathbf{k}}.$$

Thus,

$$D_{\mathbf{T}}f = \nabla f \cdot \hat{\mathbf{T}} = (-5\hat{\mathbf{i}} - \hat{\mathbf{j}} + 4\hat{\mathbf{k}}) \cdot \frac{(\hat{\mathbf{i}} - 4\hat{\mathbf{k}})}{\sqrt{17}} = -\frac{21}{\sqrt{17}}.$$

**2.** Find the equation of the tangent plane to the surface

$$x^2y^3 + e^{2(y-2)} = 13 + xz^2,$$

at the point $(-1,2,2)$. Simplify your equation as much as possible.

A vector normal to the tangent plane is

$$\begin{aligned}\nabla(x^2y^3 + e^{2(y-2)} - xz^2 - 13)_{|(-1,2,2)} &= [(2xy^3 - z^2)\hat{\mathbf{i}} + (3x^2y^2 + 2e^{2(y-2)})\hat{\mathbf{j}} - 2xz\,\hat{\mathbf{k}}]_{|(-1,2,2)} \\ &= -20\hat{\mathbf{i}} + 14\hat{\mathbf{j}} + 4\hat{\mathbf{k}}.\end{aligned}$$

So also is $-10\hat{\mathbf{i}} + 7\hat{\mathbf{j}} + 2\hat{\mathbf{k}}$. The equation of the tangent plane is

$$0 = -10(x+1) + 7(y-2) + 2(z-2) = -10x + 7y + 2z - 28.$$

**3.** You are given that $z = u^2v + v^2u$ and $u$ and $v$ are defined as functions of $x$ and $y$ by the equations

$$u^2x + v^2y = 4 + xy, \qquad uv + x^2y^2 + xyuv = 1.$$

Find $\left.\dfrac{\partial z}{\partial x}\right)_y$. Your answer may be left in terms of unsimplified determinants, but all partial derivatives must be calculated.

The chain rule gives

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x} = (2uv + v^2)\frac{\partial u}{\partial x} + (u^2 + 2uv)\frac{\partial v}{\partial x}.$$

If we set $F(x,y,u,v) = u^2x + v^2y - xy - 4$ and $G(x,y,u,v) = uv + x^2y^2 + xyuv - 1$, then

$$\frac{\partial u}{\partial x} = -\frac{\dfrac{\partial(F,G)}{\partial(x,v)}}{\dfrac{\partial(F,G)}{\partial(u,v)}} = -\frac{\begin{vmatrix} F_x & F_v \\ G_x & G_v \end{vmatrix}}{\begin{vmatrix} F_u & F_v \\ G_u & G_v \end{vmatrix}} = -\frac{\begin{vmatrix} u^2 - y & 2vy \\ 2xy^2 + yuv & u + xyu \end{vmatrix}}{\begin{vmatrix} 2ux & 2vy \\ v + xyv & u + xyu \end{vmatrix}}$$

$$\frac{\partial v}{\partial x} = -\frac{\dfrac{\partial(F,G)}{\partial(u,x)}}{\dfrac{\partial(F,G)}{\partial(u,v)}} = -\frac{\begin{vmatrix} F_u & F_x \\ G_u & G_x \end{vmatrix}}{\begin{vmatrix} F_u & F_v \\ G_u & G_v \end{vmatrix}} = -\frac{\begin{vmatrix} 2ux & u^2 - y \\ v + xyv & 2xy^2 + yuv \end{vmatrix}}{\begin{vmatrix} 2ux & 2vy \\ v + xyv & u + xyu \end{vmatrix}}.$$

**4.** Find all critical points of the function

$$f(x,y) = x^2y - 2xy^2 + 4xy.$$

Classify **ONE** of them as giving a relative maximum, a relative minimum, a saddle point, or none of these.

For critical points,

$$0 = f_x = 2xy - 2y^2 + 4y = 2y(x - y + 2), \qquad 0 = f_y = x^2 - 4xy + 4x = x(x - 4y + 4).$$

Solutions are $(0,0)$, $(-4,0)$, $(0,2)$, and $(-4/3, 2/3)$. Second derivatives are

$$f_{xx} = 2y, \qquad f_{xy} = 2x - 4y + 4, \qquad f_{yy} = -4x.$$

For the critical point $(0,0)$, $A = 0$, $B = 4$, and $C = 0$. Since $B^2 - AC = 16 > 0$, $(0,0)$ gives a saddle point.