MATH 2130 Test 2 Solutions Fall 2012 **60 minutes**

Values

6 **1.** The only point at which the curve

$$x = y^2 + z, \quad x + y = 1$$

intersects the surface

$$2y + 2z - x^2 y = 0$$

is $(0, 1, -1)$. Find the cosine of the angle between the tangent to the curve and the normal to the surface at this point.

If $F(x, y, z) = x - y^2 - z$, then a tangent vector to the curve is

$$\nabla F_{|(0,1,-1)} \times \langle 1, 1, 0\rangle = \langle 1, -2y, -1\rangle_{|(0,1,-1)} \times \langle 1, 1, 0\rangle = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & -2 & -1 \\ 1 & 1 & 0 \end{vmatrix} = \langle 1, -1, 3\rangle.$$

A vector perpendicular to the surface $2y + 2z - x^2 y = 0$ at the point $(0, 1, -1)$ is

$$\nabla(2y + 2z - x^2 y)_{|(0,1,-1)} = \langle -2xy, 2 - x^2, 2\rangle_{|(0,1,-1)} = \langle 0, 2, 2\rangle.$$

If θ is the angle between these vectors, then

$$\langle 1, -1, 3\rangle \cdot \langle 0, 2, 2\rangle = |\langle 1, -1, 3\rangle|\,|\langle 0, 2, 2\rangle| \cos\theta.$$

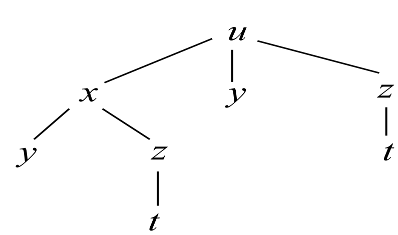
This gives

$$\cos\theta = \frac{4}{\sqrt{11}\sqrt{8}} = \frac{2}{\sqrt{22}} = \frac{\sqrt{22}}{11}.$$

6 **2.** If $u = f(x, y, z)$, $x = g(y, z)$, and $z = h(t)$, find the chain rule for $\left.\dfrac{\partial u}{\partial t}\right)_y$.

From the schematic,

$$\left.\frac{\partial u}{\partial t}\right)_y = \frac{\partial u}{\partial x}\frac{\partial x}{\partial z}\frac{dz}{dt} + \frac{\partial u}{\partial z}\frac{dz}{dt}.$$



6 **3.** Find all directions in which the rate of change of the function $f(x, y, z) = x^2yz + xy$ is equal to zero at the point $(1, -1, 2)$. Express your answer as a vector.

First we calculate the gradient of $f(x, y, z)$ at $(1, -1, 2)$,

$$\nabla f_{|(1,-1,2)} = \langle 2xyz + y, x^2z + x, x^2y\rangle_{|(1,-1,2)} = \langle -5, 3, -1\rangle.$$

If $\mathbf{v} = \langle a, b, c\rangle$ is a vector along which the rate of change of the function is zero, then

$$0 = D_{\mathbf{v}}f = \langle -5, 3, -1\rangle \cdot \frac{\langle a, b, c\rangle}{\sqrt{a^2 + b^2 + c^2}} = \frac{-5a + 3b - c}{\sqrt{a^2 + b^2 + c^2}}.$$

Thus, $c = -5a + 3b$, and required vectors are $\mathbf{v} = \langle a, b, -5a + 3b\rangle$.

11 **4.** The equations

$$u^2 + v^3 + xu^3 + 2y = 1, \quad u^3 + uy - 3ux - 3vx = 0,$$

define u and v as functions of x and y. Find $\dfrac{\partial v}{\partial y}$ when $x = 0$ and $y = 1$.

If we define $F(x, y, u, v) = u^2 + v^3 + xu^3 + 2y - 1$, and $G(x, y, u, v) = u^3 + uy - 3ux - 3vx$, then

$$\frac{\partial v}{\partial y} = -\frac{\dfrac{\partial(F,G)}{\partial(u,y)}}{\dfrac{\partial(F,G)}{\partial(u,v)}} = -\frac{\begin{vmatrix} F_u & F_y \\ G_u & G_y \end{vmatrix}}{\begin{vmatrix} F_u & F_v \\ G_u & G_v \end{vmatrix}} = -\frac{\begin{vmatrix} 2u + 3xu^2 & 2 \\ 3u^2 + y - 3x & u \end{vmatrix}}{\begin{vmatrix} 2u + 3xu^2 & 3v^2 \\ 3u^2 + y - 3x & -3x \end{vmatrix}}.$$

When $x = 0$ and $y = 1$, the equations reduce to

$$u^2 + v^3 + 2 = 1, \qquad u^3 + u = 0.$$

The only solution is $u = 0$ and $v = -1$. With these values,

$$\frac{\partial v}{\partial y} = -\frac{\begin{vmatrix} 0 & 2 \\ 1 & 0 \end{vmatrix}}{\begin{vmatrix} 0 & 3 \\ 1 & 0 \end{vmatrix}} = -\frac{2}{3}.$$

11 **5.** (a) Find all critical points for the function $f(x, y) = 4x^2 - 12xy + 9y^2$.

(b) Verify that the second derivative test fails to classify any of the critical points as yielding relative maxima, relative minima, or saddle points.

(c) Find a classification for each critical point.

(a) For critical points, we solve

$$0 = f_x = 8x - 12y, \qquad 0 = f_y = -12x + 18y.$$

Both equations give $y = 3x/2$. In other words, every point on the line $y = 3x/2$ is critical.

(b)We now calculate

$$f_{xx} = 8, \qquad f_{xy} = -12, \qquad f_{yy} = 18.$$

Since $B^2 - AC = 0$, the second derivative test fails to classify any of the critical points as yielding relative maxima, relative minima, or saddle points.

(c) If we express the function in the form $f(x, y) = (2x - 3y)^2$, we see that at every critical point, the value of the function is zero, and at every other point, the value of the function is positive. This implies that every critical point yields a relative minimum for the function.