MATH 2130 Test 2 November 17, 2020 60 minutes

4 **1.** The gradient of a funnction $f(x,y,z)$ at the point $(1,-3,4)$ is

$$\nabla f_{|(1,-3,4)} = (5,-2,6).$$

(a) Considering rates of change of $f(x,y,z)$ along all vectors with tails at $(1,-3,4)$, what is the maximum rate of change of $f(x,y,z)$?

(b) Describe in words all directions in which the rate of change of $f(x,y,z)$ is equal to zero.

(c) Is it possible to find directions at $(1,-3,4)$ in which the rate of change of $f(x,y,z)$ is equal to 9? Explain.

(a) The maximum rate of change is the length of the gradient vector,

$$|\nabla f_{|(1,-3,4)}| = \sqrt{25+4+36} = \sqrt{65}.$$

(b) In any direction perpendicular to the gradient vector, the rate of change is equal to zero.

(c) Since the maximum rate of change is $\sqrt{65}$, there is no direction in which the rate of change is equal to 9.

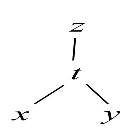
5 **2.** If $z = f(t)$, and $t = g(x,y)$, find the chain rule for $\left.\dfrac{\partial^2 z}{\partial x^2}\right)_y$.

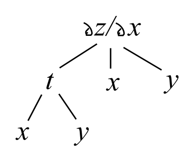
With the left schematic below,

$$\frac{\partial z}{\partial x} = \frac{dz}{dt}\frac{\partial t}{\partial x}.$$

With the right schematic,

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial t}\left(\frac{\partial z}{\partial x}\right)\frac{\partial t}{\partial x} + \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \left(\frac{d^2 z}{dt^2}\frac{\partial t}{\partial x}\right)\frac{\partial t}{\partial x} + \frac{dz}{dt}\frac{\partial^2 t}{\partial x^2}.$$





7 **3.** Find equations for the tangent line to the curve

$$x^2y^3 + xz = 1, \quad 2xz + 3yz^3 + xy = -23$$

at the point $(1, -1, 2)$.

$$\nabla(x^2y^3 + xz - 1)_{|(1,-1,2)} = (2xy^3 + z, 3x^2y^2, x)_{|(1,-1,2)} = (0, 3, 1),$$
$$\nabla(2xz + 3yz^3 + xy + 23)_{|(1,-1,2)} = (2z + y, 3z^3 + x, 2x + 9yz^2)_{|(1,-1,2)} = (3, 25, -34).$$

A tangent vector to the curve at $(1, -1, 2)$ is

$$\begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 0 & 3 & 1 \\ 3 & 25 & -34 \end{vmatrix} = (-127, 3, -9).$$

Equations for the tangent line are

$$x = 1 - 127t, \quad y = -1 + 3t, \quad z = 2 - 9t.$$

7 **4.** Find all critical points for the function

$$f(x, y) = ye^x - e^y$$

and classify them as yielding relative maxima, relative minima, saddle points, or none of these.

For critical points, we solve

$$0 = f_x = ye^x, \quad 0 = f_y = e^x - e^y \qquad \Longrightarrow \qquad x = 0, \quad y = 0.$$

The only critical point is $(0, 0)$. Since

$$f_{xx} = ye^x, \quad f_{xy} = e^x, \quad \text{and} \quad f_{yy} = -e^y,$$

we find that at $(0, 0)$, $A = 0$, $B = 1$, and $C = -1$. Since $B^2 - AC = 1 > 0$, $(0, 0)$ gives a saddle point.

7 **5.** Find the maximum value for the function

$$f(x, y) = y(1 - x - y)$$

on the region bounded by the lines

$$x + y = 1, \quad x = 0, \quad y = 0.$$

For critical points in the region, we solve

$$0 = f_x = -y, \quad 0 = f_y = 1 - 2y - x \qquad \Longrightarrow \qquad x = 1, \quad y = 0.$$

We therefore calculate $f(1, 0) = \boxed{0}$. On the boundary $x = 0$,

$$f(0, y) = y(1 - y) = F(y), \quad 0 \le y \le 1.$$

For critical values, we set

$$0 = F'(y) = 1 - 2y \quad \Longrightarrow \quad y = 1/2.$$

We evaluate

$$F(0) = \boxed{0}, \quad F(1/2) = \boxed{1/4}, \quad F(1) = \boxed{0}.$$

On the boundary $y = 0$, $f(x, 0) = 0$ for all $0 \le x \le 1$. On the boundary, $x + y = 1$, $f(x, y) = 0$ also. Thus, the maximum value is $1/4$.