MATH 2130 Test 2 Solutions

6 **1.** Find parametric equations for the normal line to the surface

$$x^3yz + xe^{yz} + z = 5$$

at the point $(1, 0, 4)$.

A normal vector to the surface is

$$\nabla(x^3yz + xe^{yz} + z - 5)_{|(1,0,4)} = (3x^2yz + e^{yz}, x^3z + xze^{yz}, x^3y + xye^{yz} + 1)_{|(1,0,4)} = (1, 8, 1).$$

Parametic equations for the normal line are therefore

$$x = 1 + t, \quad y = 8t, \quad z = 4 + t.$$

13 **2.** The following equations define u and v as functions of x and y

$$x^4u^2v + xy^3 = 1, \qquad x^3 \sin y + uv = 2.$$

Find $\partial u/\partial y$ when $x = 1$ and $y = 0$.

If we set $F(x, y, u, v) = x^4u^2v + xy^3 - 1$, and $G(x, y, u, v) = x^3 \sin y + uv - 2$, then

$$\frac{\partial u}{\partial y} = -\frac{\dfrac{\partial(F,G)}{\partial(y,v)}}{\dfrac{\partial(F,G)}{\partial(u,v)}} = -\frac{\begin{vmatrix} F_y & F_v \\ G_y & G_v \end{vmatrix}}{\begin{vmatrix} F_u & F_v \\ G_u & G_v \end{vmatrix}} = -\frac{\begin{vmatrix} 3xy^2 & x^4u^2 \\ x^3\cos y & u \end{vmatrix}}{\begin{vmatrix} 2x^4uv & x^4u^2 \\ v & u \end{vmatrix}}.$$

When $x = 1$ and $y = 0$, the equations reduce to

$$u^2v = 1, \quad uv = 2 \qquad \text{and these imply that} \qquad u = 1/2, \quad v = 4.$$

With these values,

$$\frac{\partial u}{\partial y} = -\frac{\begin{vmatrix} 0 & 1/4 \\ 1 & 1/2 \end{vmatrix}}{\begin{vmatrix} 4 & 1/4 \\ 4 & 1/2 \end{vmatrix}} = \frac{1}{4}.$$

12 **3.** Find the rate of change of the function

$$f(x,y,z) = \ln(2x - y + 3z)$$

with respect to distance along the curve

$$x^3y^2 + xyz = -1, \qquad xz^2 + yz = 2$$

in the direction of decreasing z at the point $(1,-1,2)$.

The gradient of the function at $(1,-1,2)$ is

$$\nabla(\ln(2x-y+3z))_{|(1,-1,2)} = \left(\frac{2}{2x-y+3z}, \frac{-1}{2x-y+3z}, \frac{3}{2x-y+3z}\right)_{|(1,-1,2)} = \left(\frac{2}{9}, -\frac{1}{9}, \frac{3}{9}\right).$$

A normal vector to the surface $x^3y^2 + xyz = -1$ at $(1,-1,2)$ is

$$\nabla(x^3y^2 + xyz + 1)_{|(1,-1,2)} = (3x^2y^2 + yz, 2x^3y + xz, xy)_{|(1,-1,2)} = (1,0,-1).$$

A normal vector to the surface $xz^2 + yz = 2$ at $(1,-1,2)$ is

$$\nabla(xz^2 + yz - 2)_{|(1,-1,2)} = (z^2, z, 2xz + y)_{|(1,-1,2)} = (4,2,3).$$

A vector along the tangent line to the curve at the point is

$$\mathbf{T} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & 0 & -1 \\ 4 & 2 & 3 \end{vmatrix} = (2,-7,2).$$

The rate of change in the direction of decreasing z is

$$D_{-\hat{T}}f = \nabla f \cdot (-\hat{\mathbf{T}}) = \left(\frac{2}{9}, -\frac{1}{9}, \frac{3}{9}\right) \cdot \frac{(-2,7,-2)}{\sqrt{57}} = -\frac{17}{9\sqrt{57}}.$$

6 **4.** Prove that $(1,2)$ is a critical point for the function

$$f(x,y) = |x-1| + 4(y-2)^2.$$

Determine whether this critical point yields a relative maximum, a relative minimum, a saddle point, or none of these.

For critical points of the function, we first set

$$0 = f_x = \frac{x-1}{|x-1|}, \qquad 0 = f_y = 8(y-2).$$

There are no solutions to these equations. But, f_x does not exist when $x = 1$. Hence, critical points of the function are $(1,y)$ for any y. It follows that $(1,2)$ is a critical point.

Since $f(1,2) = 0$ and $f(x,y) > 0$ for all other values of x and y, $(1,2)$ yields a relative minimum.

3 **5.** Prove that the function

$$f(x, y, z) = x^3 \cos\left(\frac{y+z}{x}\right) + xyz$$

satisfies the equation

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + z\frac{\partial f}{\partial z} = 3f(x, y, z).$$

Hint: Think before making any calculations.

Since

$$f(tx, ty, tz) = (tx)^3 \cos\left(\frac{ty+tz}{tx}\right) + (tx)(ty)(tz) = t^3\left[x^3 \cos\left(\frac{y+z}{x}\right) + xyz\right] = t^3\, f(x, y, z),$$

the function is homogeneous of degree 3. Because partial derivatives are continuous when $x \neq 0$, Euler's theorem gives the result.