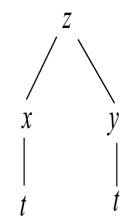
MATH 2130 Test 2 November 1, 2022 60 minutes

6 **1.** If $z = x^2 + y^2 - xy$, $x = 3t^2 \sin t$, and $y = t^3 \cos 2t$, find dz/dt.

Using the schematic to the right,

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$
$$= (2x - y)(6t \sin t + 3t^2 \cos t) + (2y - x)(3t^2 \cos 2t - 2t^3 \sin 2t).$$



12 **2.** The following equations define s and t as functions of x and y.

$$x^3s^2 + 2xty + 5sy = 5, \quad xt + xys^2 + y^3 + t^3 = 9.$$

Find $\partial s/\partial y$ when $x = 0$ and $y = 1$.

When we let $F(x, y, s, t) = x^3s^2 + 2xty + 5sy - 5$ and $G(x, y, s, t) = xt + xys^2 + y^3 + t^3 - 9$,

$$\frac{\partial s}{\partial y} = -\frac{\dfrac{\partial(F, G)}{\partial(y, t)}}{\dfrac{\partial(F, G)}{\partial(s, t)}} = -\frac{\begin{vmatrix} F_y & F_t \\ G_y & G_t \end{vmatrix}}{\begin{vmatrix} F_s & F_t \\ G_s & G_t \end{vmatrix}} = -\frac{\begin{vmatrix} 2xt + 5s & 2xy \\ xs^2 + 3y^2 & x + 3t^2 \end{vmatrix}}{\begin{vmatrix} 2x^3s + 5y & 2xy \\ 2xys & x + 3t^2 \end{vmatrix}}.$$

When $x = 0$ and $y = 1$,

$$5s = 5, \quad 1 + t^3 = 9 \qquad \text{and these imply that} \qquad s = 1, \quad t = 2.$$

The partial derivative at these values of x, y, s, and t is

$$\frac{\partial s}{\partial y} = -\frac{\begin{vmatrix} 5 & 0 \\ 3 & 12 \end{vmatrix}}{\begin{vmatrix} 5 & 0 \\ 0 & 12 \end{vmatrix}} = -1.$$

12 **3.** Find the rate(s) of change of the function

$$f(x, y, z) = x^2yz^3 + z\sin(xy) + z + x$$

with respect to distance along the curve

$$x^2y^3z + xy + 5z = -5, \qquad 2x^2 + y^2 + 3z = -1$$

at the point $(1, 0, -1)$.

$$\nabla f_{|(1,0,-1)} = (2xyz^3 + yz\cos(xy) + 1, x^2z^3 + xz\cos(xy), 3x^2yz^2 + \sin(xy) + 1)_{|(1,0,-1)} = (1, -2, 1)$$

Since

$$\nabla(x^2y^3z + xy + 5z + 5)_{|(1,0,-1)} = (2xy^3z + y, 3x^2y^2z + x, x^2y^3 + 5)_{|(1,0,-1)} = (0, 1, 5)$$

and

$$\nabla(2x^2 + y^2 + 3z + 1)_{|(1,0,-1)} = (4x, 2y, 3)_{|(1,0,01)} = (4, 0, 3),$$

a tangent vector to the curve at $(1, 0, -1)$ is

$$\mathbf{T} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 0 & 1 & 5 \\ 4 & 0 & 3 \end{vmatrix} = (3, 20, -4).$$

The rate of change in this direction is

$$D_{\mathbf{T}}f = \nabla f \cdot \hat{\mathbf{T}} = (1, -2, 1) \cdot \frac{(3, 20, -4)}{\sqrt{425}} = \frac{-41}{\sqrt{425}}.$$

We could also proceed in the opposite direction in which case the rate of change is $41/\sqrt{425}$.

10 **4.** (a) Find all critical points for the function

$$f(x, y) = y^3 + 3x^2y - 6x^2 - 6y^2 + 2.$$

(b) Pick one of the critical points and classify it as giving a relative maximum, a relative minimum, a saddle point, or none of these.

For critical points we sove

$$0 = f_x = 6xy - 12x = 6x(y - 2). \qquad 0 = f_y = 3y^2 + 3x^2 - 12y.$$

The first implies that $x = 0$ or $y = 2$. When $x = 0$, the second equation becomes $0 = 3y^2 - 12y = 3y(y - 4)$ so that $y = 0$ or $y = 4$. Thus, two critical points are $(0, 0)$ and $(0, 4)$. When $y = 2$, the second equation gives $0 = 12 + 3x^2 - 24$, from which $x = \pm 2$. Thus, two more critical points are $(\pm 2, 2)$.

$$f_{xx} = 6y - 12, \qquad f_{xy} = 6x, \qquad f_{yy} = 6y - 12.$$

At the critical point $(0, 0)$, $A = -12$, $B = 0$, and $C = -12$. Hence, $B^2 - AC = -144$, and with $A < 0$, $(0, 0)$ gives a relative maximum.