MATH 2130 Test 1 June 8, 2021 60 minutes

12 **1.** Find the rates of change of the function $f(x,y,z) = xy + xz$ at the point $(1,-1,3)$ with respect to distance along the curve

$$xyz + x^2z + y = -1, \qquad 2x + y^2 + z = 6.$$

$$\nabla f_{|(1,-1,3)} = (y+z, x, x)_{|(1,-1,3)} = (2,1,1)$$

If we set $F(x,y,z) = xyz + x^2z + y + 1$ and $G(x,y,z) = 2x + y^2 + z - 6$, then

$$\nabla F_{|(1,-1,3)} = (yz + 2xz, xz + 1, xy + x^2)_{|(1,-1,3)} = (3,4,0),$$
$$\nabla G_{|(1,-1,3)} = (2, 2y, 1)_{|(1,-1,3)} = (2,-2,1).$$

A tangent vector to the curve at $(1,-1,3)$ is

$$\mathbf{T} = \nabla F \times \nabla G = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 3 & 4 & 0 \\ 2 & -2 & 1 \end{vmatrix} = (4,-3,-14).$$

A unit tangent vector is $\hat{\mathbf{T}} = \dfrac{(4,-3,-14)}{\sqrt{221}}$. The directional derivative of $f(x,y,z)$ in direction $\hat{\mathbf{T}}$ is

$$D_{\hat{\mathbf{T}}}f = \nabla f \cdot \hat{\mathbf{T}} = (2,2,1) \cdot \frac{(4,-3,-14)}{\sqrt{221}} = \frac{-9}{\sqrt{221}}.$$

Since direction along the curve was not specified, the rate of change in the opposite direction is $9/\sqrt{221}$.

10 **2.** (a) Set up, but do **NOT** evaluate, double iterated integral(s) representing the double integral of the function $f(x, y) = 3xy^2 + 2x^2y^2$ over the region bounded by the curves

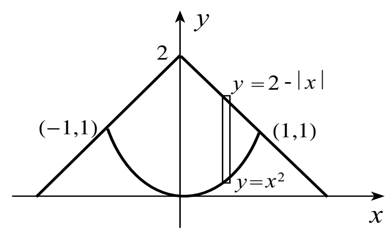
$$y = x^2, \qquad y = 2 - |x|.$$

(b) Without evaluating the double iterated integral(s) in part (a), simplify it (or them) as much as possible. In particular, your final answer should not contain absolute values.

(a) From the diagram to the right,

$$\iint_R (3xy^2 + 2x^2y^2)\, dA = \int_{-1}^{1} \int_{x^2}^{2-|x|} (3xy^2 + 2x^2y^2)\, dy\, dx.$$

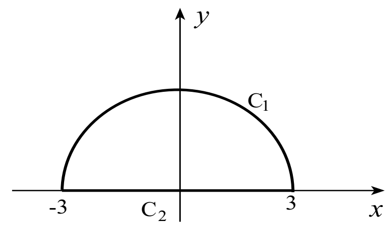
(b) Since $3xy^2$ is an odd function of x, and R is symmetric about the y-axis, this term contributes to zero the double integral, and we can therefore eliminate it. Since $2x^2y^2$ is an even function of x, we can double the integral over the right half. In other words,



$$\iint_R (3xy^2 + 2x^2y^2)\, dA = 4 \int_{0}^{1} \int_{x^2}^{2-x} x^2y^2\, dy\, dx.$$

18 **3.** Find the maximum value of the function $f(x,y) = x^2 + y^2 - xy$ on the region bounded by the curves

$$y = \sqrt{9 - x^2}, \qquad y = 0.$$



For critical points in the interior of the region, we solve $0 = f_x = 2x - y$ and $0 = f_y = 2y - x$. The only critical point is $(0,0)$ at which $f(0,0) = \boxed{0}$.
On edge C_2, $y = 0$, in which case

$$f(x,0) = g(x) = x^2, \quad -3 \le x \le 3.$$

For critical values, we solve

$$0 = g'(x) = 2x \quad \Longrightarrow \quad x = 0.$$

We evaluate

$$g(-3) = \boxed{9}, \quad g(0) = \boxed{0}, \quad g(3) = \boxed{9}.$$

On C_1, we set $x = 3\cos t$ and $y = 3\sin t$ in which case

$$f(3\cos t, 3\sin t) = h(t) = 9 - 9\cos t \sin t = 9 - \frac{9}{2}\sin 2t, \quad 0 \le t \le \pi.$$

For critical values, we solve

$$0 = h'(t) = -9\cos 2t \quad \Longrightarrow \quad 2t = \frac{\pi}{2} + n\pi \quad \Longrightarrow \quad t = \frac{\pi}{4} + \frac{n\pi}{2}.$$

Thus, $t = \pi/4$ and $t = 3\pi/4$. We evaluate

$$h(0) = \boxed{9}, \quad h(\pi/4) = \boxed{9/2}, \quad h(3\pi/4) = \boxed{27/2}, \quad h(\pi) = \boxed{9}.$$

The maximun value is $27/2$.

As an alternative on C_1, we could write

$$f(x, \sqrt{9 - x^2}) = h(x) = 9 - x\sqrt{9 - x^2}, \quad -3 \le x \le 3.$$

For critical values, we solve

$$0 = h'(x) = -\sqrt{9 - x^2} - x(1/2)(9 - x^2)^{-1/2}(-2x) = \frac{2x^2 - 9}{\sqrt{9 - x^2}}.$$

Solutions are $x = \pm 3/\sqrt{2}$. We evaluate

$$h(-3) = \boxed{9}, \quad h(3/\sqrt{2}) = \boxed{9/2}, \quad h(-3/\sqrt{2}) = \boxed{27/2}, \quad h(3) = \boxed{9}.$$