**MATH 2130 Test 2 Solutions 2012** **60 minutes**

**Values**

**6** **1.** Determine whether the limit

$$\lim_{(x,y)\to(0,1)} \frac{x^2+y^2-2y+1}{3x^2-4y^2+8y-4}$$

exists. If the limit exists, find its value; if the limit does not exist, give reasons for its nonexistence.

If we approach $(0,1)$ along straight lines $y = mx+1$, then

$$\lim_{(x,y)\to(0,1)} \frac{x^2+y^2-2y+1}{3x^2-4y^2+8y-4} = \lim_{x\to 0} \frac{x^2+(y-1)^2}{3x^2-4(y-1)^2} = \lim_{x\to 0} \frac{x^2+m^2x^2}{3x^2-4m^2x^2}$$
$$= \lim_{x\to 0} \frac{1+m^2}{3-4m^2} = \frac{1+m^2}{3-4m^2}.$$

Because this depends on $m$, the function has different limits for different modes of approach. Therefore the given limit does not exist.

**5** **2.** Are there any values of $n$ for which the function $f(x,y) = (2x+3y)^n$ is harmonic in the $xy$-plane?

The function has continuous derivatives of all orders. In addition,

$$\frac{\partial f}{\partial x} = 2n(2x+3y)^{n-1}, \qquad \frac{\partial^2 f}{\partial x^2} = 4n(n-1)(2x+3y)^{n-2},$$

$$\frac{\partial f}{\partial y} = 3n(2x+3y)^{n-1}, \qquad \frac{\partial^2 f}{\partial y^2} = 9n(n-1)(2x+3y)^{n-2},$$

The function satisfies Laplace's equation if

$$0 = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 13n(n-1)(2x+3y)^{n-2}.$$

This is true only if $n=0$ or $n=1$.

**9** **3.** The function $f(x,y,z) = x^2e^{-z}+y$ is defined at every point on the curve

$$x^2+y^2=4, \quad z=x,$$

directed counterclockwise as viewed from the origin. Find the rate of change of the function with respect to distance along the curve at the point $(2,0,2)$.

Since the origin is at the centre of the circle, we can go either direction around the curve. Since parametric equations for the curve are

$$x = 2\cos t, \quad y = 2\sin t, \quad z = 2\cos t,$$

a tangent vector to the curve at $(2,0,2)$ is

$$\mathbf{T}(0) = (-2\sin t, 2\cos t, -2\sin t)_{|t=0} = (0,2,0).$$

A unit tangent vector is $\hat{\mathbf{T}} = \hat{\mathbf{j}}$. The gradient of the function at $(2,0,2)$ is

$$\nabla f_{|(2,0,2)} = (2xe^{-z}, 1, -x^2e^{-z})_{|(2,0,2)} = (4e^{-2}, 1, -4e^{-2}).$$

Consequently,

$$D_{\mathbf{T}}f = (4e^{-2}, 1, -4e^{-2}) \cdot \hat{\mathbf{j}} = 1.$$

In the reverse direction, the rate of change is $-1$.

10 **4.** The equations

$$x^2 - y^2 = 2uv, \quad u^2 + v^2 = 2xy$$

define $x$ and $y$ as functions of $u$ and $v$. Find $\dfrac{\partial x}{\partial v}$, simplified as much as possible.

If we set $F(x,y,u,v) = x^2 - y^2 - 2uv$ and $G(x,y,u,v) = u^2 + v^2 - 2xy$, then

$$\frac{\partial x}{\partial v} = -\frac{\dfrac{\partial(F,G)}{\partial(v,y)}}{\dfrac{\partial(F,G)}{\partial(x,y)}} = -\frac{\begin{vmatrix} F_v & F_y \\ G_v & G_y \end{vmatrix}}{\begin{vmatrix} F_x & F_y \\ G_x & G_y \end{vmatrix}} = -\frac{\begin{vmatrix} -2u & -2y \\ 2v & -2x \end{vmatrix}}{\begin{vmatrix} 2x & -2y \\ -2y & -2x \end{vmatrix}}$$

$$= -\frac{4ux + 4vy}{-4x^2 - 4y^2} = \frac{ux + vy}{x^2 + y^2}.$$

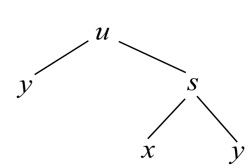
10 **5.** If $f(x)$ is a differentiable function, verify that the function $u(x,y) = y\,f(3x^2 - 4y)$ satisfies the equation

$$\frac{2y}{x}\frac{\partial u}{\partial x} + 3y\frac{\partial u}{\partial y} = 3u.$$

If we set $s = 3x^2 - 4y$, then $u = y\,f(s)$, and

$$\frac{\partial u}{\partial x} = yf'(s)(6x),$$

$$\frac{\partial u}{\partial y} = f(s) + yf'(s)(-4).$$




Consequently,

$$\frac{2y}{x}\frac{\partial u}{\partial x} + 3y\frac{\partial u}{\partial y} = \frac{2y}{x}[6xyf'(s)] + 3y[f(s) - 4yf'(s)] = 3yf(s) = 3u.$$