MATH2132 Test1 **October 8, 2020** **60 minutes**

Values

4 **1.** Determine whether the following series converges or diverges. Justify your answer.

$$\sum_{n=2}^{\infty} (-1)^n \left(\frac{n+1}{n}\right)^n$$

Since $\lim_{n\to\infty} \left(\frac{n+1}{n}\right)^n = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e$, it follows that

$$\lim_{n\to\infty} \left[(-1)^n \left(\frac{n+1}{n}\right)^n\right] \quad \text{does not exist.}$$

Hence, the series diverges by the n^{th}-term test.

8 **2.** Find the open interval of convergence for the power series

$$\sum_{n=14}^{\infty} \frac{n^2}{e^n}(x+1)^{2n}.$$

Determine, with justification, whether the series converges at its right endpoint.

If we set $y = (x+1)^2$, the series becomes $\sum_{n=14}^{\infty} \frac{n^2}{e^n} y^n$. The radius of convergence of this series is

$$R_y = \lim_{n\to\infty} \left| \frac{\frac{n^2}{e^n}}{\frac{(n+1)^2}{e^{n+1}}} \right| = e.$$

The radius of convergence of the original series is therefore $R_x = \sqrt{e}$. Its open interval of convergence is

$$|x+1| < \sqrt{e} \quad \Longrightarrow \quad -\sqrt{e} < x+1 < \sqrt{e} \quad \Longrightarrow \quad -1-\sqrt{e} < x < -1+\sqrt{e}.$$

At the right endpoint $x = -1+\sqrt{e}$, the series becomes

$$\sum_{n=14}^{\infty} \frac{n^2}{e^n}(\sqrt{e})^{2n} = \sum_{n=14}^{\infty} n^2.$$

Since the $\lim_{n\to\infty} (n^2) = \infty$, this series diverges by the n^{th}-term test.

9 **3.** Find the Taylor series about $x = 4$ for the function

$$\frac{(x-4)^2}{(x+5)^2}.$$

Write your answer in sigma notation simplified as much as possible. You must use a method that guarantees that the series converges to the function. What is the interval of convergence of the series?

$\boxed{\text{x-4}}$

$$\frac{1}{x+5} = \frac{1}{(x-4)+9} = \frac{1/9}{1+\left(\dfrac{x-4}{9}\right)} = \frac{1}{9}\sum_{n=0}^{\infty}\left[-\left(\frac{x-4}{9}\right)\right]^n = \sum_{n=0}^{\infty}\frac{(-1)^n}{9^{n+1}}(x-4)^n$$

valid for

$$\left|-\left(\frac{x-4}{9}\right)\right| < 1 \implies |x-4| < 9 \implies -9 < x-4 < 9 \implies -5 < x < 13.$$

Since the series has a positive radius of convergence, we can differentiate the series term-by-term

$$\frac{-1}{(x+5)^2} = \sum_{n=0}^{\infty}\frac{(-1)^n n}{9^{n+1}}(x-4)^{n-1}.$$

Multiplication by $-(x-4)^2$ gives

$$\frac{(x-4)^2}{(x+5)^2} = \sum_{n=0}^{\infty}\frac{(-1)^{n+1}n}{9^{n+1}}(x-4)^{n+1} = \sum_{n=2}^{\infty}\frac{(-1)^n(n-1)}{9^n}(x-4)^n.$$

Since differentiation of a series never picks up an end point of the open interval of convergence, the interval of convergence is $-5 < x < 13$.

9 **4.** Find the sum of the series

$$\sum_{n=1}^{\infty} n(-1)^n(x-1)^n.$$

The radius of convergence of the series is

$$R = \lim_{n\to\infty}\left|\frac{n(-1)^n}{(n+1)(-1)^{n+1}}\right| = 1.$$

If we set $S(x) = \sum_{n=1}^{\infty} n(-1)^n(x-1)^n$, then

$$\frac{S(x)}{x-1} = \sum_{n=1}^{\infty} n(-1)^n(x-1)^{n-1}, \qquad (x \neq 1).$$

Since the series has a positive radius of convergence, we can integrate the series term-by-term

$$\int \frac{S(x)}{x-1}\,dx = \sum_{n=1}^{\infty} (-1)^n (x-1)^n + C = \frac{-(x-1)}{1+(x-1)} + C = \frac{1-x}{x} + C.$$

Differentiation gives

$$\frac{S(x)}{x-1} = \frac{x(-1)-(1-x)}{x^2} = -\frac{1}{x^2} \qquad \Longrightarrow \qquad S(x) = \frac{1-x}{x^2}.$$

When $x = 1$, the sum of the series is zero. Since $S(1) = 0$ also, we can say that

$$S(x) = \frac{1-x}{x^2},$$

valid for $|x-1| < 1 \quad \Longrightarrow \quad 0 < x < 2.$