Solutions to Fall 2022 Exam

1. (a) Find the sum of the following series. It is not necessary to simplify your answer.

$$\sum_{n=2}^{\infty} \frac{(-1)^n 3^n}{5^{n+2}} (x+2)^{2n+1}$$

(b) What is the interval of convergence of the series? Write it in one of the forms $a < x < b$, $a \le x < b$, $a < x \le b$, or $a \le x \le b$.

(a) When we write the series in the form

$$\sum_{n=2}^{\infty} \left(\frac{x+2}{25}\right) \left[-\frac{3}{5}(x+2)^2\right]^n,$$

we see that it is geometric with common ratio $-(3/5)(x+2)^2$. The sum is

$$\frac{\left(\frac{x+2}{25}\right)\left[-\frac{3}{5}(x+2)^2\right]^2}{1+\frac{3}{5}(x+2)^2}.$$

(b) The interval of convergence is

$$\left|-\frac{3}{5}(x+2)^2\right| < 1 \quad \Longrightarrow \quad -\frac{5}{3} < (x+2)^2 < \frac{5}{3} \quad \Longrightarrow \quad (x+2)^2 < \frac{5}{3} \quad \Longrightarrow \quad -\sqrt{\frac{5}{3}} < x+2 < \sqrt{\frac{5}{3}}$$

$$\Longrightarrow \quad -2-\sqrt{\frac{5}{3}} < x < -2+\sqrt{\frac{5}{3}}.$$

2. Find the Taylor series about $x = 1$ for the function $\mathrm{Tan}^{-1}(x-1)$. Write your final answer in sigma notation, simplified as much as possible. You must use a method that guarantees that the series converges to $f(x)$. Determine the open interval of convergence of the series.

$\boxed{x-1}$ We begin with

$$\frac{1}{1+(x-1)^2} = \sum_{n=0}^{\infty} [-(x-1)^2]^n = \sum_{n=0}^{\infty} (-1)^n (x-1)^{2n},$$

valid for $|x-1|^2 < 1 \quad \Longrightarrow \quad |x-1| < 1 \quad \Longrightarrow \quad -1 < x-1 < 1 \quad \Longrightarrow \quad 0 < x < 2$. Because the radius of convergence ($R = 1$) is positive, we can integrate the series term-by-term,

$$\mathrm{Tan}^{-1}(x-1) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}(x-1)^{2n+1} + C.$$

Substitution of $x = 1$ gives $C = 0$, and therefore

$$\mathrm{Tan}^{-1}(x-1) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}(x-1)^{2n+1}.$$

The open interval of convergence is $0 < x < 2$.

3. (a) Find an expression for the maximum error when the first 4 terms of the Maclaurin series for the function

$$f(x) = \frac{1}{1+ax}, \qquad a \text{ a positive constant,}$$

are used to approximate the function on the interval $0 \le x \le 0.1$.

(b) What is the largest value of a if the maximum error must be less than or equal to 10^{-8}?

(a) When we expand the function with a geometric series,

$$f(x) = \sum_{n=0}^{\infty}(-ax)^n = \sum_{n=0}^{\infty}(-1)^n a^n x^n = 1 - ax + a^2x^2 - a^3x^3 + a^4x^4 + \cdots.$$

On the interval $0 \le x \le 0.1$, the series is alternating. Absolute values of terms are decreasing and have limit zero if $a \le 1$. Hence, the maximum error when the first four terms are used to approximate the function is

$$|a^4x^4| \le \frac{a^4}{10^4}.$$

(b) If the maximum error must be less than or equal to 10^{-8}, then

$$\frac{a^4}{10^4} \le 10^{-8} \qquad \Longrightarrow \qquad a^4 \le 10^{-4} \qquad \Longrightarrow \qquad a \le \frac{1}{10}.$$

Alternatively, the absolute value of the maximum error is

$$|R_3(0,x)| = \left|\frac{f^4(z)}{4!}x^4\right| = \frac{24a^4}{(1+az)^5}\frac{x^4}{4!} \le a^4x^4 \le \frac{a^4}{10^4}.$$

4. Find a general solution for the differential equation

$$\frac{d^3y}{dx^3} - 2\frac{d^2y}{dx^2} + 3\frac{dy}{dx} - 18y = x + 2e^x.$$

The auxiliary equation is $0 = m^3 - 2m^2 + 3m - 18 = (m-3)(m^2+m+6)$ with roots $m = 3$ and $m = (-1 \pm \sqrt{1-24})/2 = -1/2 \pm \sqrt{23}i/2$. A general solution of the associated homogeneous equation is

$$y_h(x) = C_1e^{3x} + e^{-x/2}\left(C_2\cos\frac{\sqrt{23}x}{2} + C_3\sin\frac{\sqrt{23}x}{2}\right).$$

We substitute a particular solution of the form $y_p(x) = Ax + B + Ce^x$ into the differential equation

$$(Ce^x) - 2(Ce^x) + 3(A + Ce^x) - 18(Ax + B + Ce^x) = x + 2e^x.$$

When we equate coefficients,

$$1: \quad 3A - 18B = 0, \qquad x: \quad -18A = 1, \qquad e^x: \quad C - 2C + 3C - 18C = 2.$$

These gives $A = -1/18$, $B = -1/108$, and $C = -1/8$. Thus,

$$y(x) = C_1e^{3x} + e^{-x/2}\left(C_2\cos\frac{\sqrt{23}x}{2} + C_3\sin\frac{\sqrt{23}x}{2}\right) - \frac{x}{18} - \frac{1}{108} - \frac{e^x}{8}.$$

5. What is the form of a particular solution as predicted by undetermined coefficients for the differential equation

$$D^2(D^2+1)^2(D^2+2D+5)y = -x + 6 + \cos x + 5xe^x.$$

Since roots of the auxiliary equation $m^2(m^2+1)^2(m^2+2m+5) = 0$ are

$$0,\ 0,\ \pm i,\ \pm i,\ (-2 \pm \sqrt{4-20})/2 = -1 \pm 2i,$$

a general solution of the associated homogeneous equation is

$$y_h(x) = C_1 + C_2 x + (C_3 + C_4 x)\cos x + (C_5 + C_6 x)\sin x + e^{-x}(C_7 \cos 2x + C_8 \sin 2x).$$

It follows that a particular solution must be of the form

$$y_p(x) = Ax^3 + Bx^2 + Cx^2 \cos x + Dx^2 \sin x + Exe^x + Fe^x.$$

6. A 2 kilogram mass hangs motionless on the end of a spring with constant 12 newtons per metre. It is set into vertical motion by giving it velocity 2 metres per second downward. During its subsequent motion, it is subjected to a damping force equal to 10 times its speed. Determine all times when the mass passes through its equilibrium position.

The initial-value problem for displacement $x(t)$ from its equilibrium position is

$$2\frac{d^2x}{dt^2} + 10\frac{dx}{dt} + 12x = 0, \qquad x(0) = 0, \quad x'(0) = -2.$$

The auxiliary equation is

$$0 = 2m^2 + 10m + 12 = 2(m+2)(m+3) \qquad \Longrightarrow \qquad m = -2, -3.$$

Thus, $x(t) = C_1 e^{-2t} + C_2 e^{-3t}$. The initial conditions require

$$0 = C_1 + C_2, \quad -2 = -2C_1 - 3C_2 \qquad \Longrightarrow C_1 = -2, \quad C_2 = 2.$$

Displacement is therefore $x(t) = 2(e^{-3t} - e^{-2t})$ m. The mass is at equilibrium if, and when,

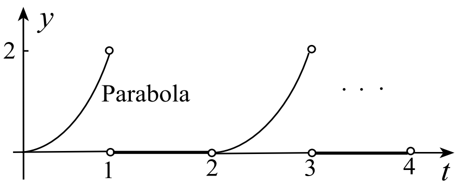
$$0 = x(t) = 2(e^{-3t} - e^{-2t}) \quad \Longrightarrow \quad e^{-3t} = e^{-2t} \qquad \Longrightarrow \qquad t = 0.$$

Since the mass starts from equilibrium, it never passes through it.

7. Find the Laplace transform for the function $f(t) = e^{2t}\sin(3t)\, h(t-6)$.

$$\begin{aligned} F(s) &= e^{-6s}\mathcal{L}\{e^{2(t+6)}\sin 3(t+6)\} = e^{-6s}e^{12}\mathcal{L}\{\sin(3t+18)\}_{|s\to s-2} \\ &= e^{-6(s-2)}\mathcal{L}\{\sin 3t \cos 18 + \cos 3t \sin 18\}_{|s\to s-2} \\ &= e^{-6(s-2)}\left[\frac{3\cos 18}{(s-2)^2+9} + \frac{\sin 18(s-2)}{(s-2)^2+9}\right] \end{aligned}$$

8. Find the Laplace transform for the periodic function in the diagram below.



$$F(s) = \frac{1}{1-e^{-2s}}\mathcal{L}\{2t^2[h(t)-h(t-1)]\} = \frac{1}{1-e^{-2s}}\left[\frac{4}{s^3} - e^{-s}\mathcal{L}\{2(t+1)^2\}\right]$$
$$= \frac{1}{1-e^{-2s}}\left[\frac{4}{s^3} - 2e^{-s}\mathcal{L}\{t^2+2t+1\}\right] = \frac{1}{1-e^{-2s}}\left[\frac{4}{s^3} - 2e^{-s}\left(\frac{2}{s^3}+\frac{2}{s^2}+\frac{1}{s}\right)\right]$$

12 **9.** Find inverse Laplace transforms for the functions

(a) $F(s) = \dfrac{s+11}{s^3+6s^2+11s}$ (b) $F(s) = \dfrac{e^{-s}}{s^2(1+e^{-3s})}$

(a) Since $F(s) = \dfrac{s+11}{s(s^2+6s+11)} = \dfrac{1}{s} - \dfrac{s+5}{s^2+6s+11}$,

$$f(t) = 1 - \mathcal{L}^{-1}\left\{\frac{(s+3)+2}{(s+3)^2+2}\right\} = 1 - e^{-3t}\mathcal{L}^{-1}\left\{\frac{s+2}{s^2+2}\right\}$$
$$= 1 - e^{-3t}\left[\cos\sqrt{2}t + \sqrt{2}\sin\sqrt{2}t\right].$$

(b) Since $F(s) = \dfrac{e^{-s}}{s^2}\displaystyle\sum_{n=0}^{\infty}(-e^{-3s})^n = \sum_{n=0}^{\infty}\frac{1}{s^2}(-1)^n e^{-(3n+1)s}$,

$$f(t) = \sum_{n=0}^{\infty}(-1)^n(t-3n-1)h(t-3n-1).$$

10. Prove that if $f(t)$ has a Laplace transform, then

$$\mathcal{L}\{f(t)h(t-a)\} = e^{-as}\mathcal{L}\{f(t+a)\}.$$

By definition, $\mathcal{L}\{f(t)h(t-a)\} = \displaystyle\int_0^{\infty} e^{-st}f(t)h(t-a)\,dt = \int_a^{\infty} e^{-st}f(t)\,dt$. If we set $u = t-a$ and $du = dt$,

$$\mathcal{L}\{f(t)h(t-a)\} = \int_0^{\infty} e^{-s(u+a)}f(u+a)\,du = e^{-as}\int_0^{\infty} e^{-su}f(u+a)\,du$$
$$= e^{-as}\int_0^{\infty} e^{-st}f(t+a)\,dt = e^{-as}\mathcal{L}\{f(t+a)\}.$$

11. A 5-kilogram mass is suspended from a spring with constant 200 newtons per metre. The mass is set into vertical motion by giving it velocity 20 centimetres per second upward. During its subsequent motion, it experiences damping equal in magnitude to 10 times its speed. If the mass is struck with a hammer at $t = 3$ seconds increasing its momentum by 5 kilograms per metre, find the position of the mass as a function of time.

The initial-value problem for displacement $x(t)$ of the mass from equilibrium is

$$5\frac{d^2x}{dt^2} + 10\frac{dx}{dt} + 200x = 5\delta(t-3), \qquad x(0) = 0, \quad x'(0) = 1/5.$$

When we divide by 5, and take Laplace transforms,

$$\left(s^2X - \frac{1}{5}\right) + 2sX + 40X = e^{-3s} \qquad \Longrightarrow \qquad X(s) = \frac{1/5 + e^{-3s}}{s^2 + 2s + 40}.$$

Displacement is therefore

$$\begin{aligned} x(t) &= \mathcal{L}^{-1}\left\{\frac{1/5 + e^{-3s}}{s^2 + 2s + 40}\right\} = \frac{1}{5}\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2 + 39} + \frac{5e^{-3s}}{(s+1)^2 + 39}\right\} \\ &= \frac{1}{5}e^{-t}\mathcal{L}^{-1}\left\{\frac{1}{s^2 + 39}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2 + 39}\right\}_{|t \to t-3} h(t-3) \\ &= \frac{1}{5\sqrt{39}}e^{-t}\sin\sqrt{39}t + \frac{1}{\sqrt{39}}e^{-(t-3)}\sin\sqrt{39}(t-3)h(t-3) \text{ m.} \end{aligned}$$

12. (a) A tank contains 1000 litres of water in which 5 kilograms of salt have been dissolved. Starting at time $t = 0$, solution with concentration 2 kilograms of salt per 100 litres of water is added to the tank at the rate of 20 millilitres per second. At the same time mixture starts being drawn from the tank at 10 millilitres per second. After 1 minute, 200 grams of salt are suddenly added to the tank, and an additional 1000 grams are added after another minute. Assume that the mixture in the tank is always sufficiently well-stirred that at any given time, concentration of salt is the same at all points in the tank, even immediately after the bulk additions. Set up, **but do NOT solve**, an initial value problem for the number of grams of salt in the tank.
(b) Can Laplace transforms be used to solve this problem? Explain.

(a) If $x(t)$ is the number of grams of salt in the tank as a function of time t in seconds,

$$\frac{dx}{dt} = \frac{2}{5} + 200\delta(t-60) + 1000\delta(t-120) - \frac{10x}{10^6 + 10t},$$

subject to $x(0) = 5000$.
(b) Laplace transforms cannot handle the last term in the differential equation.