Alternating Series

A series of constants $\sum_{n=1}^{\infty} c_n$ is said to be **alternating** if its terms are alternately positive and negative. For example, the series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

is called the **alternating harmonic series**. We know that the harmonic series which has all positive terms diverges. The partial cancelling effect of the negative terms creates a convergent series. We can prove this with the following theorem.

Theorem 1 (**Alternating Series Test**) An alternating series $\sum_{n=1}^{\infty} c_n$ converges if the sequence of absolute values of the terms $\{|c_n|\}$ is decreasing and has limit zero.

The alternating harmonic series satisfies the conditions of this theorem, the sequence $\{1/n\}$ is decreasing and has limit zero. Hence, the alternating harmonic series converges. Although it is not obvious, the sum of the series is $\ln 2$.

Example 1 Determine whether the following alternating series converge or diverge,

$$\text{(a)} \sum_{n=3}^{\infty} \frac{(-1)^n n}{2n^2+3} \qquad\qquad \text{(b)} \sum_{n=1}^{\infty} (-1)^{n+1} \frac{n+1}{5n+2}$$

Solution (a) Since the sequence $\left\{\dfrac{n}{2n^2+3}\right\}$ is decreasing and has limit zero, the alternating series test implies that the series converges.

(b) The sequence $\left\{\dfrac{n+1}{5n+2}\right\}$ is decreasing, but it has limit $1/5$, not zero. We cannot conclude by the alternating series test that the series diverges. However, we can say that $\lim_{n\to\infty}\left\{(-1)^{n+1}\dfrac{n+1}{5n+2}\right\}$ does not exist. Hence the series diverges by the n^{th}-term test.•

Example 2 Find the interval of convergence of the power series $\sum_{n=1}^{\infty} \dfrac{2^n}{n+1} x^n$.

Solution The radius of convergence of the series is

$$R = \lim_{n\to\infty} \left| \frac{\dfrac{2^n}{n+1}}{\dfrac{2^{n+1}}{n+2}} \right| = \frac{1}{2}.$$

The open interval of convergence of the series is therefore $-1/2 < x < 1/2$. At $x = 1/2$, the power series becomes

$$\sum_{n=1}^{\infty} \frac{1}{n+1} = \frac{1}{2} + \frac{1}{3} + \cdots,$$

the harmonic series, missing the first term. The series diverges. At $x = -1/2$, the power series becomes

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n+1} = -\frac{1}{2} + \frac{1}{3} - \cdots,$$

the alternating harmonic series, less the first term. The series therefore converges. The interval of convergence of the power series is therefore $-1/2 \le x < 1/2$.•

We have a formula for the sum of geometric series. Sums for other convergent series of constants can sometimes be found, especially if they relate to the exponential, sine, and cosine functions. The technique for finding sums of certain power series in Section 10.6 also lead to sums of convergent series of numbers. These techniques certainly do not find sums for all convergent series of numbers. In fact, there are many series of numbers for which we would find it impossible to find a sum. But in applications we might be satisfied with a reasonable approximation to the sum of such a series, and we therefore turn our attention to the problem of estimating the sum of a convergent series of numbers. The easiest method for estimating the sum S of a convergent series $\sum_{n=1}^{\infty} c_n$ is simply to choose the partial sum S_N for some N as an approximation; that is, truncate the series after N terms and choose

$$S \approx S_N = c_1 + c_2 + \cdots + c_N.$$

But an approximation is of value only if we can make some definitive statement about its accuracy. In truncating the series, we have neglected the infinity of terms $\sum_{n=N+1}^{\infty} c_n$, and the accuracy of the approximation is therefore determined by the size of $\sum_{n=N+1}^{\infty} c_n$; the smaller it is, the better the approximation. The problem is that we do not know the exact value of $\sum_{n=N+1}^{\infty} c_n$; if we did, there would be no need to approximate the sum of the original series in the first place. What we must do is estimate the sum $\sum_{n=N+1}^{\infty} c_n$.

It is very simple to obtain the **truncation error**, an estimate of the accuracy of a truncated alternating series $\sum c_n$ provided the sequence $\{|c_n|\}$ is decreasing with limit zero. For example, suppose $c_1 > 0$ (a similar discussion can be made when $c_1 < 0$). If $\{S_n\}$ is the sequence of partial sums of $\sum c_n$, then even partial sums can be expressed in the form

$$S_{2n} = (c_1 + c_2) + (c_3 + c_4) + \cdots + (c_{2n-1} + c_{2n}).$$

Since $\{|c_n|\}$ is decreasing ($|c_n| > |c_{n+1}|$), each term in the parentheses is positive. Consequently, the subsequence $\{S_{2n}\}$ of even partial sums of $\{S_n\}$ is increasing and approaches the sum of the series $\sum c_n$ from below (see figure). In a similar way, we can show that the subsequence $\{S_{2n-1}\}$ of odd partial sums is decreasing and approaches the sum of the series from above. It follows that the sum $\sum c_n$ must be between any two terms of the subsequences $\{S_{2n}\}$ and $\{S_{2n-1}\}$. In particular, *the sum of the alternating series must be between any two successive partial sums.* Furthermore, *when the alternating series is truncated, the maximum possible error is the next term.*

Example 3 Find a three-decimal approximation for the sum of the series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^6}$.

Solution Since the series is alternating, and absolute values of terms are decreasing with limit zero, we can say that the sum of the series lies between any two successive partial sums. We therefore calculate partial sums of the series until two successive partial sums agree to three decimals.

$$S_1 = 1, \quad S_2 = 0.984375, \quad S_3 = 0.985747, \quad S_4 = 0.985503.$$

Since S_3 and S_4 both round to 0.986, this is a three decimal approximation to the sum of the series.•

In practical situations, we often have to decide how many terms of a series to take in order to guarantee a certain degree of accuracy. Once again this is easy for alternating series whose terms satisfy the conditions of the alternating series test.

Example 4 How many terms in the series $\sum_{n=2}^{\infty}(-1)^{n+1}/(n^3 \ln n)$ ensure a truncation error of less than 10^{-5}?

Solution Because absolute values of terms are decreasing and have limit zero, the maximum error in truncating this alternating series when $n = N$ is

$$\frac{(-1)^{N+2}}{(N+1)^3 \ln (N+1)}.$$

The absolute value of this error is less than 10^{-5} when

$$\frac{1}{(N+1)^3 \ln (N+1)} < 10^{-5} \qquad \text{or}$$

$$(N+1)^3 \ln (N+1) > 10^5.$$

A calculator quickly reveals that the smallest integer for which this is valid is $N = 30$. Thus, the truncated series has the required accuracy after the 29^{th} term (the first term corresponds to $n = 2$ not $n = 1$).•

Exercises

In Exercises 1–12 determine whether the series converges or diverges.

1. $\displaystyle\sum_{n=1}^{\infty}(-1)^n \frac{n}{n^3+1}$

2. $\displaystyle\sum_{n=1}^{\infty}(-1)^n \frac{n}{n^2+1}$

3. $\displaystyle\sum_{n=1}^{\infty}(-1)^n \frac{n}{n+1}$

4. $\displaystyle\sum_{n=1}^{\infty}(-1)^n \frac{n^3}{3^n}$

5. $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{\sqrt{n}}$

6. $\displaystyle\sum_{n=1}^{\infty}(-1)^n \frac{3^n}{n^3}$

7. $\displaystyle\sum_{n=1}^{\infty}(-1)^n \frac{n}{n^2+n+1}$

8. $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1} \frac{\sqrt{3n-2}}{n}$

∗9. $\displaystyle\sum_{n=1}^{\infty}(-1)^n \left(\frac{n}{n+1}\right)^n$

∗10. $\displaystyle\sum_{n=1}^{\infty}(-1)^n \frac{\sqrt{n^2+3}}{n^2+5}$

∗11. $\displaystyle\sum_{n=2}^{\infty}(-1)^{n-1} \frac{\ln n}{n}$

∗12. $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\mathrm{Tan}^{-1} n}{n^3+5n}$

In Exercises 13–16 find the interval of convergence of the power series.

∗13. $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n}x^n$

∗14. $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n2^n}(x-1)^n$

∗15. $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n(n-1)}{n^2+1}(2x)^{2n}$

∗16. $\displaystyle\sum_{n=0}^{\infty}(-1)^{n+1}\sqrt{n+1}\, x^{3n+1}$

In Exercises 17–18 use the number of terms indicated to find an approximation to the sum of the series. In each case, obtain an estimate of the truncation error.

17. $\displaystyle\sum_{n=2}^{\infty}\frac{(-1)^{n+1}}{n^3 3^n}$ (3 terms)

18. $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{n^4}$ (20 terms)

In Exercises 19–20 find an approximation to the sum accurate to five decimals.

19. $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^{n+1}}{n^3 3^n}$

20. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^4}$

In Exercises 21–22 find the number of terms of the series that should be summed so that the error is less than 10^{-6}.

21. $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^{n+1}}{n^3 3^n}$

22. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^4}$

Answers

1. Converges **2.** Converges **3.** Diverges **4.** Converges **5.** Converges **6.** Diverges
7. Converges **8.** Converges **9.** Diverges **10.** Converges **11.** Converges **12.** Converges
13. $-1 \le x < 1$ **14.** $-1 \le x < 3$ **15.** $-1/2 \le x \le 1/2$ **16.** $-1 < x < 1$
17. -0.012710, $1/(5^3 \cdot 3^5)$ **18.** -0.947030, $1/21^4$ **19.** -0.01268 **20.** -0.94703
21. 6 **22.** 31