

CHAPTER 6

EXERCISES 6.1

1. $\mathcal{L}\{t^3 - 2t^2 + 1\} = \mathcal{L}\{t^3\} - 2\mathcal{L}\{t^2\} + \mathcal{L}\{1\} = \frac{3!}{s^4} - 2\left(\frac{2!}{s^3}\right) + \frac{1}{s} = \frac{6}{s^4} - \frac{4}{s^3} + \frac{1}{s}$
2. $\mathcal{L}\{t + e^t\} = \mathcal{L}\{t\} + \mathcal{L}\{e^t\} = \frac{1}{s^2} + \frac{1}{s-1}$
3. $\mathcal{L}\{5e^{4t}\} = 5\mathcal{L}\{e^{4t}\} = \frac{5}{s-4}$
4. $\mathcal{L}\{e^{-2t} + 2e^t\} = \mathcal{L}\{e^{-2t}\} + 2\mathcal{L}\{e^t\} = \frac{1}{s+2} + \frac{2}{s-1}$
5. $\mathcal{L}\{\sin 4t + 3 \cos 4t\} = \mathcal{L}\{\sin 4t\} + 3\mathcal{L}\{\cos 4t\} = \frac{4}{s^2+16} + \frac{3s}{s^2+16} = \frac{3s+4}{s^2+16}$
6. $\mathcal{L}\{\cos 2t - 3 \sin 4t\} = \mathcal{L}\{\cos 2t\} - 3\mathcal{L}\{\sin 4t\} = \frac{s}{s^2+4} - \frac{12}{s^2+16}$
7. $\mathcal{L}\{5t \cos 2t\} = 5\mathcal{L}\{t \cos 2t\} = \frac{5(s^2-4)}{(s^2+4)^2}$
8. $\mathcal{L}\{3t \sin 4t\} = 3\mathcal{L}\{t \sin 4t\} = 3 \left[\frac{2(4)s}{(s^2+16)^2} \right] = \frac{24s}{(s^2+16)^2}$
9. $\mathcal{L}\{5t \cos t - 2t \sin t\} = 5\mathcal{L}\{t \cos t\} - 2\mathcal{L}\{t \sin t\} = 5 \left[\frac{s^2-1}{(s^2+1)^2} \right] - 2 \left[\frac{2s}{(s^2+1)^2} \right] = \frac{5s^2-4s-5}{(s^2+1)^2}$
10. $\mathcal{L}\{3t \sin t - \cos t\} = 3\mathcal{L}\{t \sin t\} - \mathcal{L}\{\cos t\} = 3 \left[\frac{2s}{(s^2+1)^2} \right] - \frac{s}{s^2+1} = \frac{5s-s^3}{(s^2+1)^2}$
11. $\mathcal{L}^{-1} \left\{ \frac{7}{s^3} \right\} = 7\mathcal{L}^{-1} \left\{ \frac{1}{s^3} \right\} = 7 \left(\frac{t^2}{2} \right) = \frac{7t^2}{2}$
12. $\mathcal{L}^{-1} \left\{ \frac{2}{s} - \frac{3}{s^4} \right\} = 2\mathcal{L}^{-1} \left\{ \frac{1}{s} \right\} - 3\mathcal{L}^{-1} \left\{ \frac{1}{s^4} \right\} = 2(1) - 3 \left(\frac{t^3}{3!} \right) = 2 - \frac{t^3}{2}$
13. $\mathcal{L}^{-1} \left\{ \frac{1}{s+5} + \frac{4}{s^2} \right\} = \mathcal{L}^{-1} \left\{ \frac{1}{s+5} \right\} + 4\mathcal{L}^{-1} \left\{ \frac{1}{s^2} \right\} = e^{-5t} + 4t$
14. $\mathcal{L}^{-1} \left\{ \frac{3}{s-1} \right\} = 3\mathcal{L}^{-1} \left\{ \frac{1}{s-1} \right\} = 3e^t$
15. $\mathcal{L}^{-1} \left\{ \frac{s}{s^2+4} - \frac{3}{s^2+4} \right\} = \mathcal{L}^{-1} \left\{ \frac{s}{s^2+4} \right\} - 3\mathcal{L}^{-1} \left\{ \frac{1}{s^2+4} \right\} = \cos 2t - 3 \left(\frac{1}{2} \sin 2t \right)$
 $= \cos 2t - \frac{3}{2} \sin 2t$
16. $\mathcal{L}^{-1} \left\{ \frac{2s}{s^2+2} - \frac{5}{s^2+9} \right\} = 2\mathcal{L}^{-1} \left\{ \frac{s}{s^2+2} \right\} - 5\mathcal{L}^{-1} \left\{ \frac{1}{s^2+9} \right\} = 2 \cos \sqrt{2}t - 5 \left(\frac{1}{3} \sin 3t \right)$
 $= 2 \cos \sqrt{2}t - \frac{5}{3} \sin 3t$
17. $\mathcal{L}^{-1} \left\{ \frac{2s}{(s^2+2)^2} \right\} = 2\mathcal{L}^{-1} \left\{ \frac{s}{(s^2+2)^2} \right\} = 2 \left(\frac{t}{2\sqrt{2}} \sin \sqrt{2}t \right) = \frac{t}{\sqrt{2}} \sin \sqrt{2}t$
18. $\mathcal{L}^{-1} \left\{ \frac{s^2}{(s^2+9)^2} \right\} = \frac{1}{6}(\sin 3t + 3t \cos 3t) = \frac{1}{6} \sin 3t + \frac{t}{2} \cos 3t$
19. $\mathcal{L}^{-1} \left\{ \frac{3s-s^2}{(s^2+4)^2} \right\} = 3\mathcal{L}^{-1} \left\{ \frac{s}{(s^2+4)^2} \right\} - \mathcal{L}^{-1} \left\{ \frac{s^2}{(s^2+4)^2} \right\} = 3 \left(\frac{t}{4} \sin 2t \right) - \frac{1}{4}(\sin 2t + 2t \cos 2t)$
 $= \frac{1}{4}(3t-1) \sin 2t - \frac{t}{2} \cos 2t$

20. $\mathcal{L}^{-1} \left\{ \frac{s^2 - 2}{(s^2 + 3)^2} \right\} = \mathcal{L}^{-1} \left\{ \frac{s^2}{(s^2 + 3)^2} \right\} - 2\mathcal{L}^{-1} \left\{ \frac{1}{(s^2 + 3)^2} \right\}$
 $= \frac{1}{2\sqrt{3}}(\sin \sqrt{3}t + \sqrt{3}t \cos \sqrt{3}t) - \frac{2}{6\sqrt{3}}(\sin \sqrt{3}t - \sqrt{3}t \cos \sqrt{3}t)$
 $= \frac{1}{6\sqrt{3}} \sin \sqrt{3}t + \frac{5t}{6} \cos \sqrt{3}t$
21. $F(s) = \int_0^\infty e^{-st} f(t) dt = \int_3^\infty e^{-st} dt = \left\{ \frac{e^{-st}}{-s} \right\}_3^\infty = \frac{e^{-3s}}{s}$, provided $s > 0$
22. $F(s) = \int_0^\infty e^{-st} f(t) dt = \int_0^4 e^{-st} dt + \int_4^\infty 2e^{-st} dt = \left\{ \frac{e^{-st}}{-s} \right\}_0^4 + 2 \left\{ \frac{e^{-st}}{-s} \right\}_4^\infty$
 $= \left(-\frac{e^{-4s}}{s} + \frac{1}{s} \right) + \frac{2e^{-4s}}{s} = \frac{1 + e^{-4s}}{s}$, provided $s > 0$
23. $F(s) = \int_0^\infty e^{-st} f(t) dt = \int_0^2 te^{-st} dt + \int_2^\infty 2e^{-st} dt = \left\{ -\frac{t}{s}e^{-st} - \frac{1}{s^2}e^{-st} \right\}_0^2 + 2 \left\{ \frac{e^{-st}}{-s} \right\}_2^\infty$
 $= \left(-\frac{2e^{-2s}}{s} - \frac{e^{-2s}}{s^2} + \frac{1}{s^2} \right) + 2 \left(\frac{e^{-2s}}{s} \right) = \frac{1}{s^2}(1 - e^{-2s})$, provided $s > 0$
24. $F(s) = \int_0^\infty e^{-st} f(t) dt = \int_0^1 t^2 e^{-st} dt = \left\{ -\frac{t^2}{s}e^{-st} - \frac{2t}{s^2}e^{-st} - \frac{2}{s^3}e^{-st} \right\}_0^1$
 $= \frac{2}{s^3} - \frac{e^{-s}}{s^3}(s^2 + 2s + 2)$, provided $s > 0$
25. $F(s) = \int_0^\infty e^{-st} f(t) dt = \int_1^\infty t^2 e^{-st} dt = \left\{ -\frac{t^2}{s}e^{-st} - \frac{2t}{s^2}e^{-st} - \frac{2}{s^3}e^{-st} \right\}_1^\infty$
 $= \frac{e^{-s}}{s^3}(s^2 + 2s + 2)$, provided $s > 0$
26. $F(s) = \int_0^\infty e^{-st} f(t) dt = \int_1^\infty (t-1)^2 e^{-st} dt = \left\{ -\frac{(t-1)^2}{s}e^{-st} - \frac{2(t-1)}{s^2}e^{-st} - \frac{2}{s^3}e^{-st} \right\}_1^\infty$
 $= \frac{2}{s^3}e^{-s}$, provided $s > 0$
27. $F(s) = \int_0^\infty e^{-st} f(t) dt = \int_1^2 e^{-st} dt = \left\{ \frac{e^{-st}}{-s} \right\}_1^2 = \frac{e^{-s} - e^{-2s}}{s}$, provided $s > 0$
28. $F(s) = \int_0^\infty e^{-st} f(t) dt = \int_0^1 te^{-st} dt + \int_1^2 (2-t)e^{-st} dt$
 $= \left\{ -\frac{t}{s}e^{-st} - \frac{1}{s^2}e^{-st} \right\}_0^1 + \left\{ \frac{t-2}{s}e^{-st} + \frac{1}{s^2}e^{-st} \right\}_1^2 = \frac{1 - 2e^{-s} + e^{-2s}}{s^2}$, provided $s > 0$
29. $F(s) = \int_0^\infty e^{-st} f(t) dt = \int_0^1 2te^{-st} dt + \int_1^\infty te^{-st} dt$
 $= 2 \left\{ -\frac{t}{s}e^{-st} - \frac{1}{s^2}e^{-st} \right\}_0^1 + \left\{ -\frac{t}{s}e^{-st} - \frac{1}{s^2}e^{-st} \right\}_1^\infty = \frac{2}{s^2} - \frac{(s+1)e^{-s}}{s^2}$, provided $s > 0$
30. $F(s) = \int_0^\infty e^{-st} f(t) dt = \int_0^1 (1+t^2)e^{-st} dt + \int_1^\infty 2te^{-st} dt$
 $= \left\{ \frac{e^{-st}}{-s} - \frac{t^2}{s}e^{-st} - \frac{2t}{s^2}e^{-st} - \frac{2}{s^3}e^{-st} \right\}_0^1 + 2 \left\{ -\frac{t}{s}e^{-st} - \frac{1}{s^2}e^{-st} \right\}_1^\infty$
 $= \frac{1}{s} + \frac{2(1 - e^{-s})}{s^3}$, provided $s > 0$

31. $F(s) = \int_0^\infty e^{-st} f(t) dt = \int_a^\infty e^{-st} dt = \left\{ \frac{e^{-st}}{-s} \right\}_a^\infty = \frac{1}{s} e^{-as}$, provided $s > 0$
32. $F(s) = \int_0^\infty e^{-st} f(t) dt = \int_a^b e^{-st} dt = \left\{ \frac{e^{-st}}{-s} \right\}_a^b = \frac{e^{-as} - e^{-bs}}{s}$, provided $s > 0$.
33. According to equation 'The Laplace transform', the Laplace transform of $\sin at$ is

$$F(s) = \int_0^\infty e^{-st} \sin at dt.$$

Integration by parts with $u = \sin at$, $du = a \cos at dt$, $dv = e^{-st} dt$, and $v = -(1/s)e^{-st}$, gives

$$F(s) = \left\{ -\frac{1}{s} e^{-st} \sin at \right\}_0^\infty - \int_0^\infty -\frac{1}{s} e^{-st} a \cos at dt = \frac{a}{s} \int_0^\infty e^{-st} \cos at dt,$$

provided $s > 0$. A second integration by parts with $u = \cos at$, $du = -a \sin at dt$, $dv = e^{-st} dt$, and $v = -(1/s)e^{-st}$, yields

$$F(s) = \frac{a}{s} \left\{ -\frac{1}{s} e^{-st} \cos at \right\}_0^\infty - \frac{a}{s} \int_0^\infty -\frac{1}{s} e^{-st} (-a \sin at) dt = \frac{a}{s^2} - \frac{a^2}{s^2} \int_0^\infty e^{-st} \sin at dt,$$

provided once again that $s > 0$. We can therefore write that

$$F(s) = \frac{a}{s^2} - \frac{a^2}{s^2} F(s),$$

and when this equation is solved for $F(s)$, the result is $F(s) = a/(s^2 + a^2)$. A similar derivation gives the transform of $\cos at$.

34. An unpalatable thing to do would be to use the definition of the transform as an improper integral. Better would be to consider the integral

$$\int_0^\infty t e^{ati} e^{-st} dt = \int_0^\infty t e^{(-s+ai)t} dt.$$

Integration by parts with $u = t$, $du = dt$, $dv = e^{(-s+ai)t} dt$, and $v = e^{(-s+ai)t}/(-s+ai)$, gives

$$\int_0^\infty t e^{ati} e^{-st} dt = \left\{ \frac{t e^{(-s+ai)t}}{-s+ai} \right\}_0^\infty - \int_0^\infty \frac{e^{(-s+ai)t}}{-s+ai} dt.$$

Now,

$$\lim_{t \rightarrow \infty} \left[\frac{t e^{(-s+ai)t}}{-s+ai} \right] = \lim_{t \rightarrow \infty} \left[\frac{t e^{-st} (\cos at + i \sin at)}{-s+ai} \right] = 0,$$

provided $s > 0$. Hence,

$$\int_0^\infty t e^{ati} e^{-st} dt = - \left\{ \frac{e^{(-s+ai)t}}{(-s+ai)^2} \right\}_0^\infty = \frac{1}{(-s+ai)^2} = \frac{(s+ai)^2}{(s-ai)^2(s+ai)^2} = \frac{s^2 - a^2 + 2asi}{(s^2 + a^2)^2}.$$

When we take real and imaginary parts,

$$\int_0^\infty t \cos at e^{-st} dt = \frac{s^2 - a^2}{(s^2 + a^2)^2}, \quad \int_0^\infty t \sin at e^{-st} dt = \frac{2as}{s^2 + a^2};$$

that is,

$$\mathcal{L}\{t \cos at\} = \frac{s^2 - a^2}{(s^2 + a^2)^2}, \quad \mathcal{L}\{t \sin at\} = \frac{2as}{(s^2 + a^2)^2}.$$

35. If we set $u = \sqrt{t}$, or, $t = u^2$ in $F(s) = \int_0^\infty \frac{1}{\sqrt{t}} e^{-st} dt$, we find

$$F(s) = \int_0^{\infty} \frac{1}{u} e^{-su^2} (2u \, du) = 2 \int_0^{\infty} e^{-su^2} \, du.$$

We now set $v = \sqrt{s}u$, in which case

$$F(s) = 2 \int_0^{\infty} e^{-v^2} \left(\frac{dv}{\sqrt{s}} \right) = \frac{2}{\sqrt{s}} \int_0^{\infty} e^{-v^2} \, dv = \frac{2}{\sqrt{s}} \left(\frac{\sqrt{\pi}}{2} \right) = \sqrt{\frac{\pi}{s}}.$$

36. Yes, they are of exponential order $O(e^{0t})$.
 37. No. The function e^{t^2} is continuous, but not of exponential order.
 38. When we take the Laplace transform of both sides of the equation and take the transform of the series term-by-term, we get

$$\mathcal{L}\{\sin at\} = \sum_{n=0}^{\infty} \frac{(-1)^n a^{2n+1}}{(2n+1)!} \mathcal{L}\{t^{2n+1}\} = \sum_{n=0}^{\infty} \frac{(-1)^n a^{2n+1}}{(2n+1)!} \left[\frac{(2n+1)!}{s^{2n+2}} \right] = \sum_{n=0}^{\infty} \frac{a}{s^2} \left(-\frac{a^2}{s^2} \right)^n.$$

This is a geometric series with common ratio $-a^2/s^2$, and therefore the sum is

$$\mathcal{L}\{\sin at\} = \frac{a/s^2}{1 + a^2/s^2} = \frac{a}{s^2 + a^2}.$$

39. The Maclaurin series for $\cos at$ is

$$\cos at = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} (at)^{2n}.$$

When we take the Laplace transform of both sides of the equation and take the transform of the series term-by-term, we get

$$\mathcal{L}\{\cos at\} = \sum_{n=0}^{\infty} \frac{(-1)^n a^{2n}}{(2n)!} \mathcal{L}\{t^{2n}\} = \sum_{n=0}^{\infty} \frac{(-1)^n a^{2n}}{(2n)!} \left[\frac{(2n)!}{s^{2n+1}} \right] = \sum_{n=0}^{\infty} \frac{1}{s} \left(-\frac{a^2}{s^2} \right)^n.$$

This is a geometric series with common ratio $-a^2/s^2$, and therefore the sum is

$$\mathcal{L}\{\cos at\} = \frac{1/s}{1 + a^2/s^2} = \frac{s}{s^2 + a^2}.$$

40. When we take the Laplace transform of both sides of the equation and take the transform of the series term-by-term, we get

$$\mathcal{L}\{J_0(t)\} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n}(n!)^2} \mathcal{L}\{t^{2n}\} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n}(n!)^2} \left[\frac{(2n)!}{s^{2n+1}} \right] = \sum_{n=0}^{\infty} \frac{(-1)^n (2n)!}{2^{2n} (n!)^2} \left(\frac{1}{s^{2n+1}} \right).$$

We can use the binomial expansion to write

$$\begin{aligned} \frac{1}{\sqrt{1+s^2}} &= \frac{1}{s\sqrt{1+1/s^2}} = \frac{1}{s} \left(1 + \frac{1}{s^2} \right)^{-1/2} \\ &= \frac{1}{s} \left[1 + \frac{-1/2}{s^2} + \frac{(-1/2)(-3/2)}{2!s^4} + \frac{(-1/2)(-3/2)(-5/2)}{3!s^6} + \dots \right] \\ &= \frac{1}{s} \left[1 - \frac{1}{2s^2} + \frac{1 \cdot 3}{2^2 2!s^4} - \frac{1 \cdot 3 \cdot 5}{2^3 3!s^6} + \dots \right] \\ &= \frac{1}{s} \left[1 + \sum_{n=1}^{\infty} \frac{(-1)^n [1 \cdot 3 \cdot 5 \cdots (2n-1)]}{2^n n! s^{2n}} \right] \\ &= \frac{1}{s} \left[1 + \sum_{n=1}^{\infty} \frac{(-1)^n [1 \cdot 2 \cdot 3 \cdots (2n-1)(2n)]}{2^n n! [2 \cdot 4 \cdot 6 \cdots (2n)] s^{2n}} \right] = \sum_{n=0}^{\infty} \frac{(-1)^n (2n)!}{2^{2n} (n!)^2 s^{2n+1}}. \end{aligned}$$

Hence, $\mathcal{L}\{J_0(t)\} = \frac{1}{\sqrt{1+s^2}}$.

41. When we expand $\sin at$ in its Maclaurin series, and take Laplace transforms term-by-term

$$\mathcal{L}\left\{\frac{\sin at}{t}\right\} = \mathcal{L}\left\{\sum_{n=0}^{\infty} \frac{(-1)^n a^{2n+1}}{(2n+1)!} t^{2n}\right\} = \sum_{n=0}^{\infty} \frac{(-1)^n a^{2n+1}}{(2n+1)!} \left[\frac{(2n)!}{s^{2n+1}}\right] = \sum_{n=0}^{\infty} \frac{(-1)^n a^{2n+1}}{(2n+1)s^{2n+1}}.$$

If we expand $1/(a^2 + s^2)$ in a geometric series, we obtain

$$\frac{1}{a^2 + s^2} = \frac{1/s^2}{1 + a^2/s^2} = \frac{1}{s^2} \sum_{n=0}^{\infty} \left(-\frac{a^2}{s^2}\right)^n = \sum_{n=0}^{\infty} \frac{(-1)^n a^{2n}}{s^{2n+2}}.$$

Term-by-term integration with respect to a gives

$$\frac{1}{s} \text{Tan}^{-1}\left(\frac{a}{s}\right) + C = \sum_{n=0}^{\infty} \frac{(-1)^n a^{2n+1}}{(2n+1)s^{2n+2}}.$$

Substitution of $a = 0$ gives $C = 0$, and therefore

$$\text{Tan}^{-1}\left(\frac{a}{s}\right) = \sum_{n=0}^{\infty} \frac{(-1)^n a^{2n+1}}{(2n+1)s^{2n+1}}.$$

Thus, $\mathcal{L}\left\{\frac{\sin at}{t}\right\} = \text{Tan}^{-1}\left(\frac{a}{s}\right)$.

42. When the exponentials are expanded in Maclaurin series,

$$\mathcal{L}\left\{\frac{e^{bt} - e^{at}}{t}\right\} = \mathcal{L}\left\{\sum_{n=1}^{\infty} \frac{b^n - a^n}{n!} t^{n-1}\right\} = \sum_{n=1}^{\infty} \frac{b^n - a^n}{n!} \left[\frac{(n-1)!}{s^n}\right] = \sum_{n=1}^{\infty} \frac{b^n - a^n}{ns^n}.$$

Expansion of $1/(s-a)$ in a geometric series leads to

$$\frac{1}{s-a} = \frac{1/s}{1-a/s} = \frac{1}{s} \sum_{n=0}^{\infty} \frac{a^n}{s^n} = \sum_{n=0}^{\infty} \frac{a^n}{s^{n+1}}.$$

Term-by-term integration with respect to a gives

$$-\ln|s-a| + C = \sum_{n=0}^{\infty} \frac{a^{n+1}}{(n+1)s^{n+1}}.$$

Evaluation at $a = 0$ implies that $C = \ln|s|$. It now follows that for $s > b > a$,

$$\begin{aligned} \ln\left(\frac{s-a}{s-b}\right) &= \ln(s-a) - \ln(s-b) = \left[\ln|s| - \sum_{n=0}^{\infty} \frac{a^{n+1}}{(n+1)s^{n+1}}\right] - \left[\ln|s| - \sum_{n=0}^{\infty} \frac{b^{n+1}}{(n+1)s^{n+1}}\right] \\ &= \sum_{n=0}^{\infty} \frac{b^{n+1} - a^{n+1}}{(n+1)s^{n+1}} = \sum_{n=1}^{\infty} \frac{b^n - a^n}{ns^n}. \end{aligned}$$

Hence, $\mathcal{L}\left\{\frac{e^{bt} - e^{at}}{t}\right\} = \ln\left(\frac{s-a}{s-b}\right)$.

43. (a) To prove this recursive formula, we use integration by parts, and to facilitate this we change the variable of integration,

$$\Gamma(t+1) = \int_0^{\infty} e^{-x} x^t dx.$$

If we set $u = x^t$, $dv = e^{-x} dx$, $du = tx^{t-1}$, and $v = -e^{-x}$,

$$\Gamma(t+1) = \{-x^t e^{-x}\}_0^\infty - \int_0^\infty -tx^{t-1}e^{-x} dx = t \int_0^\infty e^{-x}x^{t-1} dx = t\Gamma(t).$$

When t is an integer n , and we iterate the recursive formula,

$$\Gamma(n+1) = n\Gamma(n) = \cdots = n(n-1)(n-2)\cdots(1)\Gamma(1) = n! \int_0^\infty e^{-x} dx = n! \{-e^{-x}\}_0^\infty = n!.$$

(b) If we set $u = st$ in the definition of $\mathcal{L}\{t^r\}$,

$$\mathcal{L}\{t^r\} = \int_0^\infty e^{-st}t^r dt = \int_0^\infty e^{-u} \left(\frac{u}{s}\right)^r \left(\frac{du}{s}\right) = \frac{1}{s^{r+1}} \int_0^\infty e^{-u}u^r du = \frac{\Gamma(r+1)}{s^{r+1}}.$$

44. If e^{t^2} is $O(e^{\alpha t})$ for some α , there must exist constants M and T such that for all $t > T$, the function e^{t^2} would have to be less than $Me^{\alpha t} = e^{\alpha t + \ln M}$. This would mean that t^2 would have to be less than $\alpha t + \ln M$ for all $t > T$. But this is impossible. No matter what the value of α , t^2 will eventually be larger than $\alpha t + \ln M$.
45. (a) Since $f(t)$ is bounded, it is of exponential order (Exercise 'boundeD').
 (b) $f'(t) = 2te^{t^2} \cos(e^{t^2})$ For any value T , there are values of $t > T$ for which $\cos(e^{t^2}) = 1$, and for these values of t , the values of $f'(t)$ are equal to $2te^{t^2}$. Since e^{t^2} is not of exponential order, it follows that $f'(t)$ cannot be of exponential order.

EXERCISES 6.2

1. Since $f(t) = h(t-3)$, $F(s) = \mathcal{L}\{h(t-3)\} = \frac{e^{-3s}}{s}$.

2. Since $f(t) = [h(t) - h(t-4)] + 2h(t-4) = 1 + h(t-4)$,

$$F(s) = \mathcal{L}\{1 + h(t-4)\} = \frac{1}{s} + \frac{e^{-4s}}{s} = \frac{1 + e^{-4s}}{s}.$$

3. Since $f(t) = t[h(t) - h(t-2)] + 2h(t-2) = t + (2-t)h(t-2)$,

$$F(s) = \mathcal{L}\{t + (2-t)h(t-2)\} = \frac{1}{s^2} + e^{-2s}\mathcal{L}\{2 - (t+2)\} = \frac{1}{s^2} + e^{-2s}\mathcal{L}\{-t\} = \frac{1}{s^2} - \frac{e^{-2s}}{s^2} = \frac{1 - e^{-2s}}{s^2}.$$

4. Since $f(t) = t^2[h(t) - h(t-1)] = t^2 - t^2h(t-1)$,

$$\begin{aligned} F(s) &= \mathcal{L}\{t^2 - t^2h(t-1)\} = \frac{2}{s^3} - e^{-s}\mathcal{L}\{(t+1)^2\} = \frac{2}{s^3} - e^{-s}\mathcal{L}\{t^2 + 2t + 1\} \\ &= \frac{2}{s^3} - e^{-s}\left(\frac{2}{s^3} + \frac{2}{s^2} + \frac{1}{s}\right) = \frac{2}{s^3} - \frac{e^{-s}(s^2 + 2s + 2)}{s^3}. \end{aligned}$$

5. Since $f(t) = t^2h(t-1)$,

$$F(s) = \mathcal{L}\{t^2h(t-1)\} = e^{-s}\mathcal{L}\{(t+1)^2\} = e^{-s}\mathcal{L}\{t^2 + 2t + 1\} = e^{-s}\left(\frac{2}{s^3} + \frac{2}{s^2} + \frac{1}{s}\right) = \frac{e^{-s}(s^2 + 2s + 2)}{s^3}.$$

6. Since $f(t) = (t-1)^2h(t-1)$,

$$F(s) = \mathcal{L}\{(t-1)^2h(t-1)\} = e^{-s}\mathcal{L}\{(t+1-1)^2\} = e^{-s}\mathcal{L}\{t^2\} = \frac{2e^{-s}}{s^3}.$$

7. Since $f(t) = h(t-1) - h(t-2)$,

$$F(s) = \mathcal{L}\{h(t-1) - h(t-2)\} = \frac{e^{-s}}{s} - \frac{e^{-2s}}{s} = \frac{e^{-s} - e^{-2s}}{s}.$$

8. Since $f(t) = t[h(t) - h(t-1)] + (2-t)[h(t-1) - h(t-2)] = t + (2-2t)h(t-1) + (t-2)h(t-2)$,

$$\begin{aligned} F(s) &= \mathcal{L}\{t + (2-2t)h(t-1) + (t-2)h(t-2)\} = \frac{1}{s^2} + e^{-s}\mathcal{L}\{2 - 2(t+1)\} + e^{-2s}\mathcal{L}\{(t+2) - 2\} \\ &= \frac{1}{s^2} + e^{-s}\mathcal{L}\{-2t\} + e^{-2s}\mathcal{L}\{t\} = \frac{1}{s^2} - \frac{2}{s^2}e^{-s} + \frac{1}{s^2}e^{-2s} = \frac{1 - 2e^{-s} + e^{-2s}}{s^2}. \end{aligned}$$

9. Since $f(t) = 2t[h(t) - h(t-1)] + th(t-1) = 2t - th(t-1)$,

$$F(s) = \mathcal{L}\{2t - th(t-1)\} = \frac{2}{s^2} - e^{-s}\mathcal{L}\{t+1\} = \frac{2}{s^2} - e^{-s}\left(\frac{1}{s^2} + \frac{1}{s}\right) = \frac{2}{s^2} - \frac{(s+1)e^{-s}}{s^2}.$$

10. Since $f(t) = (1+t^2)[h(t) - h(t-1)] + 2th(t-1) = 1 + t^2 + (2t-1-t^2)h(t-1)$,

$$\begin{aligned} F(s) &= \mathcal{L}\{1 + t^2 + (2t-1-t^2)h(t-1)\} = \frac{1}{s} + \frac{2}{s^3} + e^{-s}\mathcal{L}\{2(t+1) - 1 - (t+1)^2\} \\ &= \frac{1}{s} + \frac{2}{s^3} + e^{-s}\mathcal{L}\{-t^2\} = \frac{1}{s} + \frac{2}{s^3} - \frac{2}{s^3}e^{-s} = \frac{1}{s} + \frac{2(1-e^{-s})}{s^3}. \end{aligned}$$

11. Since $f(t) = h(t-a)$, $F(s) = \mathcal{L}\{h(t-a)\} = \frac{e^{-as}}{s}$.

12. Since $f(t) = h(t-a) - h(t-b)$, $F(s) = \mathcal{L}\{h(t-a) - h(t-b)\} = \frac{e^{-as}}{s} - \frac{e^{-bs}}{s} = \frac{e^{-as} - e^{-bs}}{s}$.

13. Since $f(t) = 2(1-t)[h(t) - h(t-1)] = 2 - 2t + (2t-2)h(t-1)$,

$$\begin{aligned} F(s) &= \mathcal{L}\{2 - 2t + (2t-2)h(t-1)\} = \frac{2}{s} - \frac{2}{s^2} + e^{-s}\mathcal{L}\{2(t+1) - 2\} = \frac{2}{s} - \frac{2}{s^2} + e^{-s}\mathcal{L}\{2t\} \\ &= \frac{2}{s} - \frac{2}{s^2} + \frac{2e^{-s}}{s^2} = \frac{2}{s} + \frac{2(e^{-s}-1)}{s^2}. \end{aligned}$$

14. Since $f(t) = 2[h(t) - h(t-1)] + [h(t-1) - h(t-2)] + (t-2)h(t-2) = 2 - h(t-1) + (t-3)h(t-2)$,

$$\begin{aligned} F(s) &= \mathcal{L}\{2 - h(t-1) + (t-3)h(t-2)\} = \frac{2}{s} - \frac{e^{-s}}{s} + e^{-2s}\mathcal{L}\{(t+2) - 3\} = \frac{2}{s} - \frac{e^{-s}}{s} + e^{-2s}\mathcal{L}\{t-1\} \\ &= \frac{2}{s} - \frac{e^{-s}}{s} + e^{-2s}\left(\frac{1}{s^2} - \frac{1}{s}\right) = \frac{2 - e^{-s}}{s} + \frac{(1-s)e^{-2s}}{s^2}. \end{aligned}$$

15. Since $f(t) = 4(1-t^2)[h(t) - h(t-1)] = 4 - 4t^2 + 4(t^2-1)h(t-1)$,

$$\begin{aligned} F(s) &= \mathcal{L}\{4 - 4t^2 + 4(t^2-1)h(t-1)\} = \frac{4}{s} - \frac{8}{s^3} + 4e^{-s}\mathcal{L}\{(t+1)^2 - 1\} = \frac{4}{s} - \frac{8}{s^3} + 4e^{-s}\mathcal{L}\{t^2 + 2t\} \\ &= \frac{4}{s} - \frac{8}{s^3} + 4e^{-s}\left(\frac{2}{s^3} + \frac{2}{s^2}\right) = \frac{4(s^2-2)}{s^3} + \frac{8(s+1)e^{-s}}{s^3}. \end{aligned}$$

16. Since $f(t) = (1-t)[h(t) - h(t-1)] + (t-1)^2[h(t-1) - h(t-2)] = 1-t + (t^2-t)h(t-1) - (t-1)^2h(t-2)$,

$$\begin{aligned} F(s) &= \mathcal{L}\{1-t + (t^2-t)h(t-1) - (t-1)^2h(t-2)\} \\ &= \frac{1}{s} - \frac{1}{s^2} + e^{-s}\mathcal{L}\{(t+1)^2 - (t+1)\} - e^{-2s}\mathcal{L}\{(t+2-1)^2\} \\ &= \frac{1}{s} - \frac{1}{s^2} + e^{-s}\mathcal{L}\{t^2 + t\} - e^{-2s}\mathcal{L}\{t^2 + 2t + 1\} \\ &= \frac{1}{s} - \frac{1}{s^2} + e^{-s}\left(\frac{2}{s^3} + \frac{1}{s^2}\right) - e^{-2s}\left(\frac{2}{s^3} + \frac{2}{s^2} + \frac{1}{s}\right) \\ &= \frac{s-1}{s^2} + \frac{(s+2)e^{-s}}{s^3} - \frac{(s^2+2s+2)e^{-2s}}{s^3}. \end{aligned}$$

17. Since $f(t) = \sin t[h(t) - h(t-2\pi)] = \sin t - \sin t h(t-2\pi)$,

$$\begin{aligned} F(s) &= \mathcal{L}\{\sin t - \sin t h(t-2\pi)\} = \frac{1}{s^2+1} - e^{-2\pi s}\mathcal{L}\{\sin(t+2\pi)\} = \frac{1}{s^2+1} - e^{-2\pi s}\mathcal{L}\{\sin t\} \\ &= \frac{1}{s^2+1} - \frac{e^{-2\pi s}}{s^2+1} = \frac{1 - e^{-2\pi s}}{s^2+1}. \end{aligned}$$

18. Since $f(t) = \sin t h(t-2\pi)$,

$$F(s) = \mathcal{L}\{\sin t h(t-2\pi)\} = e^{-2\pi s}\mathcal{L}\{\sin(t+2\pi)\} = e^{-2\pi s}\mathcal{L}\{\sin t\} = \frac{e^{-2\pi s}}{s^2+1}.$$

19. Since $f(t) = [h(t) - h(t-\pi)] + \sin t h(t-\pi) = 1 - h(t-\pi) + \sin t h(t-\pi)$,

$$\begin{aligned} F(s) &= \mathcal{L}\{1 - h(t-\pi) + \sin t h(t-\pi)\} = \frac{1}{s} - \frac{e^{-\pi s}}{s} + e^{-\pi s}\mathcal{L}\{\sin(t+\pi)\} \\ &= \frac{1}{s} - \frac{e^{-\pi s}}{s} + e^{-\pi s}\mathcal{L}\{-\sin t\} = \frac{1 - e^{-\pi s}}{s} - \frac{e^{-\pi s}}{s^2+1}. \end{aligned}$$

20. Since $f(t) = 2e^{-t}[h(t) - h(t-\ln 2)] + h(t-\ln 2) = 2e^{-t} + (1-2e^{-t})h(t-\ln 2)$,

$$\begin{aligned} F(s) &= \mathcal{L}\{2e^{-t} + (1-2e^{-t})h(t-\ln 2)\} = \frac{2}{s+1} + e^{-s\ln 2}\mathcal{L}\{1-2e^{-(t+\ln 2)}\} \\ &= \frac{2}{s+1} + e^{-s\ln 2}\mathcal{L}\{1-e^{-t}\} = \frac{2}{s+1} + e^{-s\ln 2}\left(\frac{1}{s} - \frac{1}{s+1}\right) = \frac{2}{s+1} + \frac{e^{-s\ln 2}}{s(s+1)}. \end{aligned}$$

21. $\mathcal{L}\{t^3 e^{-5t}\} = \mathcal{L}\{t^3\}_{|s \rightarrow s+5} = \left(\frac{3!}{s^4}\right)_{|s \rightarrow s+5} = \frac{6}{(s+5)^4}$
22. $\mathcal{L}\{t^2 e^{3t}\} = \mathcal{L}\{t^2\}_{|s \rightarrow s-3} = \left(\frac{2}{s^3}\right)_{|s \rightarrow s-3} = \frac{2}{(s-3)^3}$
23. $\mathcal{L}\{4te^{-t} - 2e^{-3t}\} = 4\mathcal{L}\{t\}_{|s \rightarrow s+1} - 2\mathcal{L}\{e^{-3t}\} = 4\left(\frac{1}{s^2}\right)_{|s \rightarrow s+1} - \frac{2}{s+3} = \frac{4}{(s+1)^2} - \frac{2}{s+3}$
24. $\mathcal{L}\{5e^{at} - 5e^{-at}\} = \frac{5}{s-a} - \frac{5}{s+a} = \frac{10a}{s^2 - a^2}$
25. $\mathcal{L}\{e^t \sin 2t + e^{-t} \cos t\} = \mathcal{L}\{\sin 2t\}_{|s \rightarrow s-1} + \mathcal{L}\{\cos t\}_{|s \rightarrow s+1} = \left(\frac{2}{s^2+4}\right)_{|s \rightarrow s-1} + \left(\frac{s}{s^2+1}\right)_{|s \rightarrow s+1}$
 $= \frac{2}{(s-1)^2+4} + \frac{s+1}{(s+1)^2+1}$
26. $\mathcal{L}\{2e^{-3t} \sin 3t + 4e^{3t} \cos 3t\} = 2\mathcal{L}\{\sin 3t\}_{|s \rightarrow s+3} + 4\mathcal{L}\{\cos 3t\}_{|s \rightarrow s-3}$
 $= 2\left(\frac{3}{s^2+9}\right)_{|s \rightarrow s+3} + 4\left(\frac{s}{s^2+9}\right)_{|s \rightarrow s-3} = \frac{6}{(s+3)^2+9} + \frac{4(s-3)}{(s-3)^2+9}$
27. $\mathcal{L}\{te^t \cos 2t\} = \mathcal{L}\{t \cos 2t\}_{|s \rightarrow s-1} = \left[\frac{s^2-4}{(s^2+4)^2}\right]_{|s \rightarrow s-1} = \frac{(s-1)^2-4}{[(s-1)^2+4]^2} = \frac{s^2-2s-3}{(s^2-2s+5)^2}$
28. $\mathcal{L}\{te^{-2t} \sin t\} = \mathcal{L}\{t \sin t\}_{|s \rightarrow s+2} = \left[\frac{2s}{(s^2+1)^2}\right]_{|s \rightarrow s+2} = \frac{2(s+2)}{[(s+2)^2+1]^2} = \frac{2(s+2)}{(s^2+4s+5)^2}$
29. $\mathcal{L}\{2e^t(\cos t + \sin t)\} = 2\mathcal{L}\{\cos t + \sin t\}_{|s \rightarrow s-1} = 2\left(\frac{s}{s^2+1} + \frac{1}{s^2+1}\right)_{|s \rightarrow s-1}$
 $= 2\left[\frac{(s-1)+1}{(s-1)^2+1}\right] = \frac{2s}{s^2-2s+2}$
30. $\mathcal{L}\{(t-1)e^{2-3t} \sin 4t\} = e^2 \mathcal{L}\{t \sin 4t - \sin 4t\}_{|s \rightarrow s+3} = e^2 \left[\frac{8s}{(s^2+16)^2} - \frac{4}{s^2+16}\right]_{|s \rightarrow s+3}$
 $= e^2 \left[\frac{8s-4s^2-64}{(s^2+16)^2}\right]_{|s \rightarrow s+3} = \frac{e^2[8(s+3)-4(s+3)^2-64]}{[(s+3)^2+16]^2} = \frac{-4e^2(s^2+4s+19)}{(s^2+6s+25)^2}$

31. We calculate

$$\mathcal{L}\{t^2 e^{ati}\} = \mathcal{L}\{t^2\}_{|s \rightarrow s-ai} = \left(\frac{2}{s^3}\right)_{|s \rightarrow s-ai} = \frac{2}{(s-ai)^3}.$$

We display real and imaginary parts of both sides of the equation,

$$\mathcal{L}\{t^2(\cos at + i \sin at)\} = \frac{2(s+ai)^3}{(s-ai)^3(s+ai)^3} = \frac{2(s^3 + 3as^2i - 3a^2s - a^3i)}{(s^2+a^2)^3}.$$

When we take real parts, $\mathcal{L}\{t^2 \cos at\} = \frac{2(s^3 - 3a^2s)}{(s^2+a^2)^3}$.

32. We calculate

$$\mathcal{L}\{t^2 e^{ati}\} = \mathcal{L}\{t^2\}_{|s \rightarrow s-ai} = \left(\frac{2}{s^3}\right)_{|s \rightarrow s-ai} = \frac{2}{(s-ai)^3}.$$

We display real and imaginary parts of both sides of the equation,

$$\mathcal{L}\{t^2(\cos at + i \sin at)\} = \frac{2(s+ai)^3}{(s-ai)^3(s+ai)^3} = \frac{2(s^3 + 3as^2i - 3a^2s - a^3i)}{(s^2+a^2)^3}.$$

When we take imaginary parts, $\mathcal{L}\{t^2 \sin at\} = \frac{2(3as^2 - a^3)}{(s^2+a^2)^3}$.

33. $\mathcal{L}\{(t-2)^2 h(t-2)\} = e^{-2s} \mathcal{L}\{(t+2-2)^2\} = e^{-2s} \mathcal{L}\{t^2\} = \frac{2e^{-2s}}{s^3}$
34. $\mathcal{L}\{\sin 3(t-4)h(t-4)\} = e^{-4s} \mathcal{L}\{\sin 3(t+4-4)\} = e^{-4s} \mathcal{L}\{\sin 3t\} = \frac{3e^{-4s}}{s^2+9}$
35. $\mathcal{L}\{t h(t-1)\} = e^{-s} \mathcal{L}\{t+1\} = e^{-s} \left(\frac{1}{s^2} + \frac{1}{s} \right) = \frac{(s+1)e^{-s}}{s^2}$
36. $\mathcal{L}\{(t+5)h(t-3)\} = e^{-3s} \mathcal{L}\{(t+3+5)\} = e^{-3s} \mathcal{L}\{t+8\} = e^{-3s} \left(\frac{1}{s^2} + \frac{8}{s} \right) = \frac{(8s+1)e^{-3s}}{s^2}$
37. $\mathcal{L}\{(t^2+2)h(t-1)\} = e^{-s} \mathcal{L}\{(t+1)^2+2\} = e^{-s} \mathcal{L}\{t^2+2t+3\} = e^{-s} \left(\frac{2}{s^3} + \frac{2}{s^2} + \frac{3}{s} \right)$
 $= \frac{(2+2s+3s^2)e^{-s}}{s^3}$
38. $\mathcal{L}\{\cos t h(t-\pi)\} = e^{-\pi s} \mathcal{L}\{\cos(t+\pi)\} = e^{-\pi s} \mathcal{L}\{-\cos t\} = \frac{-se^{-\pi s}}{s^2+1}$
39. $\mathcal{L}\{\cos t h(t-2)\} = e^{-2s} \mathcal{L}\{\cos(t+2)\} = e^{-2s} \mathcal{L}\{\cos 2 \cos t - \sin 2 \sin t\}$
 $= e^{-2s} \left(\frac{s \cos 2}{s^2+1} - \frac{\sin 2}{s^2+1} \right) = \frac{e^{-2s}(s \cos 2 - \sin 2)}{s^2+1}$
40. $\mathcal{L}\{e^t h(t-4)\} = e^{-4s} \mathcal{L}\{e^{t+4}\} = e^4 e^{-4s} \mathcal{L}\{e^t\} = \frac{e^{4-4s}}{s-1}$
41. $\mathcal{L}\{t^2 e^t h(t-3)\} = e^{-3s} \mathcal{L}\{(t+3)^2 e^{t+3}\} = e^{-3s} e^3 \mathcal{L}\{(t^2+6t+9)e^t\} = e^{3-3s} \mathcal{L}\{t^2+6t+9\}_{|s \rightarrow s-1}$
 $= e^{3-3s} \left(\frac{2}{s^3} + \frac{6}{s^2} + \frac{9}{s} \right)_{|s \rightarrow s-1} = e^{3-3s} \left[\frac{2}{(s-1)^3} + \frac{6}{(s-1)^2} + \frac{9}{s-1} \right]$
 $= \frac{(9s^2-12s+5)e^{3-3s}}{(s-1)^3}$
42. $\mathcal{L}\{e^t \cos 2t h(t-1)\} = e^{-s} \mathcal{L}\{e^{t+1} \cos 2(t+1)\} = e^{-s} e \mathcal{L}\{\cos 2(t+1)\}_{|s \rightarrow s-1}$
 $= e^{1-s} \mathcal{L}\{\cos 2 \cos 2t - \sin 2 \sin 2t\}_{|s \rightarrow s-1} = e^{1-s} \left(\frac{s \cos 2}{s^2+4} - \frac{2 \sin 2}{s^2+4} \right)_{|s \rightarrow s-1}$
 $= e^{1-s} \left[\frac{(s-1) \cos 2}{(s-1)^2+4} - \frac{2 \sin 2}{(s-1)^2+4} \right] = \frac{e^{1-s}[s \cos 2 - (\cos 2 + 2 \sin 2)]}{s^2-2s+5}$
43. $F(s) = \frac{1}{1-e^{-as}} \mathcal{L}\{t[h(t)-h(t-a)]\} = \frac{1}{1-e^{-as}} \mathcal{L}\{t-t h(t-a)\} = \frac{1}{1-e^{-as}} \left[\frac{1}{s^2} - e^{-as} \mathcal{L}\{t+a\} \right]$
 $= \frac{1}{1-e^{-as}} \left[\frac{1}{s^2} - e^{-as} \left(\frac{1}{s^2} + \frac{a}{s} \right) \right] = \frac{1}{1-e^{-as}} \left[\frac{1-e^{-as}}{s^2} - \frac{ae^{-as}}{s} \right] = \frac{1}{s^2} - \frac{ae^{-as}}{s(1-e^{-as})}$
44. $F(s) = \frac{1}{1-e^{-2as}} \mathcal{L}\{[h(t)-h(t-a)] - [h(t-a)-h(t-2a)]\} = \frac{1}{1-e^{-2as}} \mathcal{L}\{1-2h(t-a)+h(t-2a)\}$
 $= \frac{1}{1-e^{-2as}} \left(\frac{1}{s} - \frac{2e^{-as}}{s} + \frac{e^{-2as}}{s} \right) = \frac{(1-e^{-as})^2}{s(1+e^{-as})(1-e^{-as})} = \frac{1-e^{-as}}{s(1+e^{-as})}$
45. $F(s) = \frac{1}{1-e^{-\pi s/a}} \mathcal{L}\{\sin at[h(t)-h(t-\pi/a)]\} = \frac{1}{1-e^{-\pi s/a}} \mathcal{L}\{\sin at - \sin at h(t-\pi/a)\}$
 $= \frac{1}{1-e^{-\pi s/a}} \left[\frac{a}{s^2+a^2} - e^{-\pi s/a} \mathcal{L}\{\sin a(t+\pi/a)\} \right] = \frac{1}{1-e^{-\pi s/a}} \left[\frac{a}{s^2+a^2} - e^{-\pi s/a} \mathcal{L}\{-\sin at\} \right]$
 $= \frac{1}{1-e^{-\pi s/a}} \left[\frac{a}{s^2+a^2} + \frac{ae^{-\pi s/a}}{s^2+a^2} \right] = \frac{a(1+e^{-\pi s/a})}{(s^2+a^2)(1-e^{-\pi s/a})}$
46. $F(s) = \frac{1}{1-e^{-2as}} \mathcal{L}\{t[h(t)-h(t-a)] + (2a-t)[h(t-a)-h(t-2a)]\}$
 $= \frac{1}{1-e^{-2as}} \mathcal{L}\{t + (2a-2t)h(t-a) + (t-2a)h(t-2a)\}$

- $$= \frac{1}{1 - e^{-2as}} \left[\frac{1}{s^2} + e^{-as} \mathcal{L}\{2a - 2(t+a)\} + e^{-2as} \mathcal{L}\{t + 2a - 2a\} \right]$$
- $$= \frac{1}{1 - e^{-2as}} \left[\frac{1}{s^2} - \frac{2e^{-as}}{s^2} + \frac{e^{-2as}}{s^2} \right] = \frac{(1 - e^{-as})^2}{s^2(1 + e^{-as})(1 - e^{-as})} = \frac{1 - e^{-as}}{s^2(1 + e^{-as})}$$
47. $F(s) = \frac{1}{1 - e^{-2as}} \mathcal{L}\{h(t) - h(t-a)\} = \frac{1}{1 - e^{-2as}} \mathcal{L}\{1 - h(t-a)\} = \frac{1}{1 - e^{-2as}} \left(\frac{1}{s} - \frac{e^{-as}}{s} \right)$
- $$= \frac{1 - e^{-as}}{s(1 + e^{-as})(1 - e^{-as})} = \frac{1}{s(1 + e^{-as})}$$
48. $\mathcal{L}^{-1} \left\{ \frac{1}{s^2 - 2s + 5} \right\} = \mathcal{L}^{-1} \left\{ \frac{1}{(s-1)^2 + 4} \right\} = e^t \mathcal{L}^{-1} \left\{ \frac{1}{s^2 + 4} \right\} = \frac{1}{2} e^t \sin 2t$
49. $\mathcal{L}^{-1} \left\{ \frac{s}{s^2 + 4s + 1} \right\} = \mathcal{L}^{-1} \left\{ \frac{(s+2) - 2}{(s+2)^2 - 3} \right\} = e^{-2t} \mathcal{L}^{-1} \left\{ \frac{s-2}{s^2 - 3} \right\}$
- $$= e^{-2t} \mathcal{L}^{-1} \left\{ \frac{(\sqrt{3}+2)/(2\sqrt{3})}{s + \sqrt{3}} + \frac{(\sqrt{3}-2)/(2\sqrt{3})}{s - \sqrt{3}} \right\}$$
- $$= e^{-2t} \left[\left(\frac{1}{2} + \frac{1}{\sqrt{3}} \right) e^{-\sqrt{3}t} + \left(\frac{1}{2} - \frac{1}{\sqrt{3}} \right) e^{\sqrt{3}t} \right]$$
- $$= \left(\frac{1}{2} + \frac{1}{\sqrt{3}} \right) e^{-(2+\sqrt{3})t} + \left(\frac{1}{2} - \frac{1}{\sqrt{3}} \right) e^{(-2+\sqrt{3})t}$$
50. $\mathcal{L}^{-1} \left\{ \frac{e^{-2s}}{s^2} \right\} = \mathcal{L}^{-1} \left\{ \frac{1}{s^2} \right\} \Big|_{t \rightarrow t-2} \quad h(t-2) = \{t\} \Big|_{t \rightarrow t-2} h(t-2) = (t-2)h(t-2)$
51. $\mathcal{L}^{-1} \left\{ \frac{e^{-3s}}{s^2 + 1} \right\} = \mathcal{L}^{-1} \left\{ \frac{1}{s^2 + 1} \right\} \Big|_{t \rightarrow t-3} \quad h(t-3) = \{\sin t\} \Big|_{t \rightarrow t-3} h(t-3) = \sin(t-3)h(t-3)$
52. $\mathcal{L}^{-1} \left\{ \frac{se^{-5s}}{s^2 + 2} \right\} = \mathcal{L}^{-1} \left\{ \frac{s}{s^2 + 2} \right\} \Big|_{t \rightarrow t-5} \quad h(t-5) = \{\cos \sqrt{2}t\} \Big|_{t \rightarrow t-5} h(t-5) = \cos \sqrt{2}(t-5)h(t-5)$
53. $\mathcal{L}^{-1} \left\{ \frac{se^{-s}}{(s^2 + 4)^2} \right\} = \mathcal{L}^{-1} \left\{ \frac{s}{(s^2 + 4)^2} \right\} \Big|_{t \rightarrow t-1} \quad h(t-1) = \left\{ \frac{t}{4} \sin 2t \right\} \Big|_{t \rightarrow t-1}$
- $$= \frac{1}{4} (t-1) \sin 2(t-1) h(t-1)$$
54. $\mathcal{L}^{-1} \left\{ \frac{1}{4s^2 - 6s - 5} \right\} = \frac{1}{4} \mathcal{L}^{-1} \left\{ \frac{1}{s^2 - 3s/2 - 5/4} \right\} = \frac{1}{4} \mathcal{L}^{-1} \left\{ \frac{1}{(s-3/4)^2 - 29/16} \right\}$
- $$= \frac{1}{4} e^{3t/4} \mathcal{L}^{-1} \left\{ \frac{1}{s^2 - 29/16} \right\} = \frac{1}{4} e^{3t/4} \mathcal{L}^{-1} \left\{ \frac{-2/\sqrt{29}}{s + \sqrt{29}/4} + \frac{2/\sqrt{29}}{s - \sqrt{29}/4} \right\}$$
- $$= \frac{1}{2\sqrt{29}} e^{3t/4} (-e^{-\sqrt{29}t/4} + e^{\sqrt{29}t/4}) = \frac{\sqrt{29}}{58} [e^{(3+\sqrt{29})t/4} - e^{(3-\sqrt{29})t/4}]$$
55. $\mathcal{L}^{-1} \left\{ \frac{s}{s^2 - 3s + 2} \right\} = \mathcal{L}^{-1} \left\{ \frac{2}{s-2} - \frac{1}{s-1} \right\} = 2e^{2t} - e^t$
56. $\mathcal{L}^{-1} \left\{ \frac{4s+1}{(s^2+s)(4s^2-1)} \right\} = \frac{1}{4} \mathcal{L}^{-1} \left\{ \frac{4s+1}{s(s+1)(s+1/2)(s-1/2)} \right\}$
- $$= \frac{1}{4} \mathcal{L}^{-1} \left\{ -\frac{4}{s} + \frac{4}{s+1} - \frac{4}{s+1/2} + \frac{4}{s-1/2} \right\} = -1 + e^{-t} - e^{-t/2} + e^{t/2}$$
57. $\mathcal{L}^{-1} \left\{ \frac{e^{-3s}}{s+5} \right\} = \mathcal{L}^{-1} \left\{ \frac{1}{s+5} \right\} \Big|_{t \rightarrow t-3} \quad h(t-3) = \{e^{-5t}\} \Big|_{t \rightarrow t-3} h(t-3) = e^{-5(t-3)}h(t-3)$
58. $\mathcal{L}^{-1} \left\{ \frac{e^{-2s}}{s^2 + 3s + 2} \right\} = \mathcal{L}^{-1} \left\{ \frac{1}{s^2 + 3s + 2} \right\} \Big|_{t \rightarrow t-2} \quad h(t-2) = \mathcal{L}^{-1} \left\{ \frac{1}{s+1} - \frac{1}{s+2} \right\} \Big|_{t \rightarrow t-2} \quad h(t-2)$

- $$\begin{aligned}
&= \{e^{-t} - e^{-2t}\}_{|t \rightarrow t-2} h(t-2) = [e^{-(t-2)} - e^{-2(t-2)}]h(t-2) \\
&= [e^{2-t} - e^{2(2-t)}]h(t-2) \\
59. \quad \mathcal{L}^{-1} \left\{ \frac{1}{s^3 + 1} \right\} &= \mathcal{L}^{-1} \left\{ \frac{1}{(s+1)(s^2 - s + 1)} \right\} = \mathcal{L}^{-1} \left\{ \frac{1/3}{s+1} + \frac{2/3 - s/3}{s^2 - s + 1} \right\} \\
&= \frac{1}{3} \mathcal{L}^{-1} \left\{ \frac{1}{s+1} + \frac{-(s-1/2) + 3/2}{(s-1/2)^2 + 3/4} \right\} = \frac{1}{3} e^{-t} + \frac{1}{3} e^{t/2} \mathcal{L}^{-1} \left\{ \frac{3/2 - s}{s^2 + 3/4} \right\} \\
&= \frac{1}{3} e^{-t} + \frac{1}{3} e^{t/2} \left(\sqrt{3} \sin \frac{\sqrt{3}t}{2} - \cos \frac{\sqrt{3}t}{2} \right) \\
60. \quad \mathcal{L}^{-1} \left\{ \frac{5s-2}{3s^2+4s+8} \right\} &= \frac{1}{3} \mathcal{L}^{-1} \left\{ \frac{5s-2}{s^2+4s/3+8/3} \right\} = \frac{1}{3} \mathcal{L}^{-1} \left\{ \frac{5(s+2/3) - 16/3}{(s+2/3)^2 + 20/9} \right\} \\
&= \frac{1}{3} e^{-2t/3} \mathcal{L}^{-1} \left\{ \frac{5s-16/3}{s^2+20/9} \right\} = \frac{1}{3} e^{-2t/3} \left(5 \cos \frac{2\sqrt{5}t}{3} - \frac{8}{\sqrt{5}} \sin \frac{2\sqrt{5}t}{3} \right) \\
61. \quad \text{Since } \mathcal{L}^{-1} \left\{ \frac{1}{s(s^2+1)} \right\} &= \mathcal{L}^{-1} \left\{ \frac{1}{s} - \frac{s}{s^2+1} \right\} = 1 - \cos t, \\
\mathcal{L}^{-1} \left\{ \frac{e^{-s}(1-e^{-s})}{s(s^2+1)} \right\} &= \mathcal{L}^{-1} \left\{ \frac{e^{-s} - e^{-2s}}{s(s^2+1)} \right\} = [1 - \cos(t-1)]h(t-1) - [1 - \cos(t-2)]h(t-2). \\
62. \quad \mathcal{L}^{-1} \left\{ \frac{s}{(s+1)^5} \right\} &= \mathcal{L}^{-1} \left\{ \frac{(s+1) - 1}{(s+1)^5} \right\} = \mathcal{L}^{-1} \left\{ \frac{1}{(s+1)^4} - \frac{1}{(s+1)^5} \right\} \\
&= e^{-t} \mathcal{L}^{-1} \left\{ \frac{1}{s^4} - \frac{1}{s^5} \right\} = e^{-t} \left(\frac{t^3}{3!} - \frac{t^4}{4!} \right) = \frac{t^3(4-t)e^{-t}}{24} \\
63. \quad \mathcal{L}^{-1} \left\{ \frac{s^2+2s+3}{(s^2+2s+2)(s^2+2s+5)} \right\} &= \mathcal{L}^{-1} \left\{ \frac{1/3}{s^2+2s+2} + \frac{2/3}{s^2+2s+5} \right\} \\
&= \frac{1}{3} \mathcal{L}^{-1} \left\{ \frac{1}{(s+1)^2+1} + \frac{2}{(s+1)^2+4} \right\} \\
&= \frac{1}{3} e^{-t} \mathcal{L}^{-1} \left\{ \frac{1}{s^2+1} + \frac{2}{s^2+4} \right\} = \frac{1}{3} e^{-t} (\sin t + \sin 2t) \\
64. \quad \mathcal{L}^{-1} \left\{ \frac{s^2}{(s^2-4)^2} \right\} &= \mathcal{L}^{-1} \left\{ \frac{1/8}{s-2} + \frac{1/4}{(s-2)^2} - \frac{1/8}{s+2} + \frac{1/4}{(s+2)^2} \right\} \\
&= \frac{1}{8} e^{2t} \mathcal{L}^{-1} \left\{ \frac{1}{s} + \frac{2}{s^2} \right\} + \frac{1}{8} e^{-2t} \mathcal{L}^{-1} \left\{ \frac{2}{s^2} - \frac{1}{s} \right\} = \frac{1}{8} e^{2t} (1+2t) + \frac{1}{8} e^{-2t} (2t-1)
\end{aligned}$$

65. We use geometric series to write $F(s)$ in the form

$$F(s) = \frac{1}{s} \sum_{n=0}^{\infty} (e^{-s})^n = \sum_{n=0}^{\infty} \frac{e^{-ns}}{s}.$$

We can now take inverse transforms,

$$f(t) = \sum_{n=0}^{\infty} h(t-n).$$

Except for values at the integers, this is the floor function $[t]$.

66. We use geometric series to write $F(s)$ in the form

$$F(s) = \frac{1}{s} \sum_{n=0}^{\infty} (-e^{-s})^n = \sum_{n=0}^{\infty} (-1)^n \frac{e^{-ns}}{s}.$$

We can now take inverse transforms,

$$f(t) = \sum_{n=0}^{\infty} (-1)^n h(t-n).$$

67. We use geometric series to write $F(s)$ in the form

$$F(s) = \frac{1}{s^2+4} \sum_{n=0}^{\infty} (e^{-3s})^n = \sum_{n=0}^{\infty} \frac{e^{-3ns}}{s^2+4}.$$

We can now take inverse transforms,

$$f(t) = \frac{1}{2} \sum_{n=0}^{\infty} \sin 2(t-3n)h(t-3n).$$

68. We use geometric series to write $F(s)$ in the form

$$F(s) = \left(\frac{1/5}{s} - \frac{s/5}{s^2+5} \right) \sum_{n=0}^{\infty} (e^{-2s})^n = \frac{1}{5} \sum_{n=0}^{\infty} \left(\frac{1}{s} - \frac{s}{s^2+5} \right) e^{-2ns}.$$

We can now take inverse transforms,

$$f(t) = \frac{1}{5} \sum_{n=0}^{\infty} \left[1 - \cos \sqrt{5}(t-2n) \right] h(t-2n).$$

69. We use geometric series to write $F(s)$ in the form

$$F(s) = \left(\frac{1/2}{s^2} + \frac{1/2}{s^2+2} \right) e^{-2s} \sum_{n=0}^{\infty} (-e^{-s})^n = \frac{1}{2} \sum_{n=0}^{\infty} (-1)^n \left(\frac{1}{s^2} + \frac{1}{s^2+2} \right) e^{-(n+2)s}.$$

We can now take inverse transforms,

$$f(t) = \frac{1}{2} \sum_{n=0}^{\infty} (-1)^n \left[(t-n-2) + \frac{1}{\sqrt{2}} \sin \sqrt{2}(t-n-2) \right] h(t-n-2).$$

70. (a) Using property ‘First shifting property’b,

$$f(t) = \mathcal{L}^{-1} \left\{ \frac{(s+1)+1}{(s+1)^2+4} \right\} = e^{-t} \mathcal{L}^{-1} \left\{ \frac{s+1}{s^2+4} \right\} = e^{-t} \left(\cos 2t + \frac{1}{2} \sin 2t \right).$$

(b) Since $s^2 + 2s + 5 = 0$ has roots $s = \frac{-2 \pm \sqrt{4-20}}{2} = -1 \pm 2i$, the quadratic factors into $(s+1-2i)(s+1+2i)$, and the partial fraction decomposition of $F(s)$ is of the form

$$\frac{s+2}{s^2+2s+5} = \frac{A}{s+1-2i} + \frac{B}{s+1+2i}.$$

Constants A and B can be obtained with the “cover up” method in Appendix D,

$$A = \frac{-1+2i+2}{4i} = \frac{2-i}{4}, \quad B = \frac{-1-2i+2}{-4i} = \frac{2+i}{4}.$$

Thus,

$$\frac{s+2}{s^2+2s+5} = \frac{1}{4} \left(\frac{2-i}{s+1-2i} + \frac{2+i}{s+1+2i} \right),$$

and

$$\begin{aligned} f(t) &= \frac{1}{4} \left[(2-i)e^{(-1+2i)t} + (2+i)e^{-(1+2i)t} \right] \\ &= \frac{e^{-t}}{4} [(2-i)(\cos 2t + i \sin 2t) + (2+i)(\cos 2t - i \sin 2t)] = \frac{e^{-t}}{4} (4 \cos 2t + 2 \sin 2t). \end{aligned}$$

71. The given property can also be written in terms of inverse transforms as

$$\mathcal{L}^{-1} \left\{ \frac{d}{ds} F(s) \right\} = -t \mathcal{L}^{-1} \{ F(s) \}.$$

When we apply this to

$$\frac{d}{ds} \left[\frac{1}{(s^2 + a^2)^n} \right] = \frac{-2ns}{(s^2 + a^2)^{n+1}},$$

we get

$$\mathcal{L}^{-1} \left\{ \frac{-2ns}{(s^2 + a^2)^{n+1}} \right\} = -t \mathcal{L}^{-1} \left\{ \frac{1}{(s^2 + a^2)^n} \right\}.$$

Thus,

$$\mathcal{L}^{-1} \left\{ \frac{s}{(s^2 + a^2)^{n+1}} \right\} = \frac{t}{2n} \mathcal{L}^{-1} \left\{ \frac{1}{(s^2 + a^2)^n} \right\}.$$

To prove the second formula, we begin with

$$\begin{aligned} \frac{d}{ds} \left[\frac{s}{(s^2 + a^2)^n} \right] &= \frac{1}{(s^2 + a^2)^n} - \frac{2ns^2}{(s^2 + a^2)^{n+1}} = \frac{1}{(s^2 + a^2)^n} - \frac{2n(s^2 + a^2) - 2na^2}{(s^2 + a^2)^{n+1}} \\ &= \frac{1}{(s^2 + a^2)^n} - \frac{2n}{(s^2 + a^2)^n} + \frac{2na^2}{(s^2 + a^2)^{n+1}}. \end{aligned}$$

If we now take inverse transforms,

$$-t \mathcal{L}^{-1} \left\{ \frac{s}{(s^2 + a^2)^n} \right\} = (1 - 2n) \mathcal{L}^{-1} \left\{ \frac{1}{(s^2 + a^2)^n} \right\} + 2na^2 \mathcal{L}^{-1} \left\{ \frac{1}{(s^2 + a^2)^{n+1}} \right\}.$$

Thus,

$$\mathcal{L}^{-1} \left\{ \frac{1}{(s^2 + a^2)^{n+1}} \right\} = \frac{-t}{2na^2} \mathcal{L}^{-1} \left\{ \frac{s}{(s^2 + a^2)^n} \right\} + \frac{2n-1}{2na^2} \mathcal{L}^{-1} \left\{ \frac{1}{(s^2 + a^2)^n} \right\}.$$

72. For line 5 in the table,

$$\mathcal{L}^{-1} \left\{ \frac{2as}{(s^2 + a^2)^2} \right\} = 2a \left[\frac{t}{2} \mathcal{L}^{-1} \left\{ \frac{1}{s^2 + a^2} \right\} \right] = at \left(\frac{1}{a} \sin at \right) = t \sin at.$$

For line 6,

$$\begin{aligned} \mathcal{L}^{-1} \left\{ \frac{s^2 - a^2}{(s^2 + a^2)^2} \right\} &= \mathcal{L}^{-1} \left\{ \frac{(s^2 + a^2) - 2a^2}{(s^2 + a^2)^2} \right\} = \mathcal{L}^{-1} \left\{ \frac{1}{s^2 + a^2} \right\} - 2a^2 \mathcal{L}^{-1} \left\{ \frac{1}{(s^2 + a^2)^2} \right\} \\ &= \frac{1}{a} \sin at - 2a^2 \left[\frac{-t}{2a^2} \mathcal{L}^{-1} \left\{ \frac{s}{s^2 + a^2} \right\} + \frac{1}{2a^2} \mathcal{L}^{-1} \left\{ \frac{1}{s^2 + a^2} \right\} \right] \\ &= \frac{1}{a} \sin at + t \cos at - \frac{1}{a} \sin at = t \cos at. \end{aligned}$$

For line 7,

$$\mathcal{L}^{-1} \left\{ \frac{2a^3}{(s^2 + a^2)^2} \right\} = 2a^3 \left[\frac{-t}{2a^2} \mathcal{L}^{-1} \left\{ \frac{s}{s^2 + a^2} \right\} + \frac{1}{2a^2} \mathcal{L}^{-1} \left\{ \frac{1}{s^2 + a^2} \right\} \right] = -at \cos at + \sin at,$$

For line 8,

$$\begin{aligned} \mathcal{L}^{-1} \left\{ \frac{2as^2}{(s^2 + a^2)^2} \right\} &= 2a \mathcal{L}^{-1} \left\{ \frac{(s^2 + a^2) - a^2}{(s^2 + a^2)^2} \right\} = 2a \mathcal{L}^{-1} \left\{ \frac{1}{s^2 + a^2} \right\} - 2a^3 \mathcal{L}^{-1} \left\{ \frac{1}{(s^2 + a^2)^2} \right\} \\ &= 2a \left(\frac{1}{a} \sin at \right) - 2a^3 \left[\frac{-t}{2a^2} \mathcal{L}^{-1} \left\{ \frac{s}{s^2 + a^2} \right\} + \frac{1}{2a^2} \mathcal{L}^{-1} \left\{ \frac{1}{s^2 + a^2} \right\} \right] \\ &= 2 \sin at + at \cos at - \sin at = \sin at + at \cos at. \end{aligned}$$

73. Using the second reduction of order formula in Exercise 'reduction of order',

$$\mathcal{L}^{-1} \left\{ \frac{1}{(s^2 + a^2)^3} \right\} = \frac{-t}{4a^2} \mathcal{L}^{-1} \left\{ \frac{s}{(s^2 + a^2)^2} \right\} + \frac{3}{4a^2} \mathcal{L}^{-1} \left\{ \frac{1}{(s^2 + a^2)^2} \right\}.$$

We can now use Table 'Table of Properties of Laplace transforms',

$$\begin{aligned} \mathcal{L}^{-1} \left\{ \frac{1}{(s^2 + a^2)^3} \right\} &= -\frac{t}{4a^2} \left[\frac{t}{2a} \sin at \right] + \frac{3}{4a^2} \left[\frac{1}{2a^3} (\sin at - at \cos at) \right] \\ &= \frac{-t^2}{8a^3} \sin at + \frac{3}{8a^5} (\sin at - at \cos at). \end{aligned}$$

74. Using the first reduction of order formula and Table 'Table of Properties of Laplace transforms',

$$\mathcal{L}^{-1} \left\{ \frac{s}{(s^2 + a^2)^3} \right\} = \frac{t}{4} \mathcal{L}^{-1} \left\{ \frac{1}{(s^2 + a^2)^2} \right\} + \frac{t}{4} \left[\frac{1}{2a^3} (\sin at - at \cos at) \right] = \frac{t}{8a^3} (\sin at - at \cos at).$$

75. When we complete the square,

$$\mathcal{L}^{-1} \left\{ \frac{1}{(s^2 - 2s + 5)^3} \right\} = \mathcal{L}^{-1} \left\{ \frac{1}{[(s-1)^2 + 4]^3} \right\} = e^t \mathcal{L}^{-1} \left\{ \frac{1}{(s^2 + 4)^3} \right\}.$$

We now use the second reduction of order formula and Table 'Table of Properties of Laplace transforms',

$$\begin{aligned} \mathcal{L}^{-1} \left\{ \frac{1}{(s^2 - 2s + 5)^3} \right\} &= e^t \left[\frac{-t}{16} \mathcal{L}^{-1} \left\{ \frac{s}{(s^2 + 4)^2} \right\} + \frac{3}{16} \mathcal{L}^{-1} \left\{ \frac{1}{(s^2 + 4)^2} \right\} \right] \\ &= e^t \left[\frac{-t}{16} \left(\frac{t}{4} \sin 2t \right) + \frac{3}{16} \left(\frac{-t}{8} \mathcal{L}^{-1} \left\{ \frac{s}{s^2 + 4} \right\} + \frac{1}{8} \mathcal{L}^{-1} \left\{ \frac{1}{s^2 + 4} \right\} \right) \right] \\ &= e^t \left[\frac{-t^2}{64} \sin 2t - \frac{3t}{128} \cos 2t + \frac{3}{256} \sin 2t \right] \\ &= \frac{e^t}{256} [(3 - 4t^2) \sin 2t - 6t \cos 2t]. \end{aligned}$$

76. When we complete the square,

$$\mathcal{L}^{-1} \left\{ \frac{s+2}{(s^2 - 4s + 13)^3} \right\} = \mathcal{L}^{-1} \left\{ \frac{(s-2)+4}{[(s-2)^2 + 9]^3} \right\} = e^{2t} \mathcal{L}^{-1} \left\{ \frac{s+4}{(s^2 + 9)^3} \right\}.$$

We now use the reduction of order formulas and Table 'Table of Properties of Laplace transforms',

$$\begin{aligned} \mathcal{L}^{-1} \left\{ \frac{1}{(s^2 - 2s + 5)^3} \right\} &= e^{2t} \left[\frac{t}{4} \mathcal{L}^{-1} \left\{ \frac{1}{(s^2 + 9)^2} \right\} - \frac{4t}{36} \mathcal{L}^{-1} \left\{ \frac{s}{(s^2 + 9)^2} \right\} + \frac{4(3)}{36} \mathcal{L}^{-1} \left\{ \frac{1}{(s^2 + 9)^2} \right\} \right] \\ &= e^{2t} \left[\frac{t}{4} \left(\frac{1}{54} \right) (\sin 3t - 3t \cos 3t) - \frac{t}{9} \left(\frac{1}{6} \right) t \sin 3t \right. \\ &\quad \left. + \frac{1}{3} \left(\frac{1}{54} \right) (\sin 3t - 3 \cos 3t) \right] \\ &= \frac{e^{2t}}{648} [(4 + 3t - 12t^2) \sin 3t - (12t + 9t^2) \cos 3t]. \end{aligned}$$

77. If $F(s) = \mathcal{L}\{f(t)\}$ for $s > \alpha$, then presumably, $f(t)$ is $O(e^{\alpha t})$. This means that $e^{at}f(t)$ is $O(e^{(\alpha+a)t})$. Thus, $F(s-a)$ is the transform of $e^{at}f(t)$ if $s > \alpha + a$.

78.
$$F(s) = \frac{1}{1 - e^{-4s}} \mathcal{L} \left\{ \frac{t^2}{4} [h(t) - h(t-1)] - \frac{1}{4} (t^2 - 4t + 2) [h(t-1) - h(t-3)] \right\}$$

$$\begin{aligned}
& + \frac{1}{4}(t-4)^2[h(t-3) - h(t-4)] \Big\} \\
= & \frac{1}{1-e^{-4s}} \mathcal{L} \left\{ \frac{t^2}{4} - \frac{1}{2}(t^2 - 2t + 1)h(t-1) + \frac{1}{2}(t^2 - 6t + 9)h(t-3) - \frac{1}{4}(t-4)^2h(t-4) \right\} \\
= & \frac{1}{4(1-e^{-4s})} \left[\frac{2}{s^3} - 2e^{-s} \mathcal{L}\{(t+1)^2 - 2(t+1) + 1\} + 2e^{-3s} \mathcal{L}\{(t+3)^2 - 6(t+3) + 9\} \right. \\
& \left. - e^{-4s} \mathcal{L}\{(t+4-4)^2\} \right] \\
= & \frac{1}{4(1-e^{-4s})} \left[\frac{2}{s^3} - 2e^{-s} \mathcal{L}\{t^2\} + 2e^{-3s} \mathcal{L}\{t^2\} - e^{-4s} \mathcal{L}\{t^2\} \right] \\
= & \frac{1}{4(1-e^{-4s})} \left[\frac{2}{s^3} + \frac{2}{s^3}(-2e^{-s} + 2e^{-3s} - e^{-4s}) \right] = \frac{1}{2s^3(1-e^{-4s})} (1 - 2e^{-s} + 2e^{-3s} - e^{-4s}) \\
= & \frac{(1-e^{-s})^2(1-e^{-2s})}{2s^3(1+e^{-2s})(1-e^{-2s})} = \frac{(1-e^{-s})^2}{2s^3(1+e^{-2s})}
\end{aligned}$$

79. If we set $u = at$ in $\mathcal{L}\{f(at)\} = \int_0^\infty f(at)e^{-st} dt$, we get

$$\mathcal{L}\{f(at)\} = \int_0^\infty f(u)e^{-s(u/a)} \left(\frac{du}{a} \right) = \frac{1}{a} \int_0^\infty f(u)e^{-(s/a)u} du = \frac{1}{a} F\left(\frac{s}{a}\right).$$

Since $F(s)$ is valid for $s > a$, it follows that $\mathcal{L}\{f(at)\}$ is valid for $s/a > \alpha$, which implies that $s > \alpha a$.

EXERCISES 6.3

1. Assuming that the solution satisfies the conditions of Corollary ‘Corollary’, we take Laplace transforms of both sides of the differential equation and use the initial conditions,

$$[s^2Y - s(1) - 0] + 3[sY - 1] - 4Y = \frac{1}{s^2} + \frac{3}{s}.$$

We solve this for the transform $Y(s)$,

$$Y(s) = \frac{3s + 1}{s^2(s^2 + 3s - 4)} + \frac{s + 3}{s^2 + 3s - 4}.$$

The inverse transform of this function is the solution of the initial-value problem

$$\begin{aligned} y(t) &= \mathcal{L}^{-1} \left\{ \frac{3s + 1}{s^2(s^2 + 3s - 4)} + \frac{s + 3}{s^2 + 3s - 4} \right\} = \mathcal{L}^{-1} \left\{ -\frac{15/16}{s} - \frac{1/4}{s^2} + \frac{27/80}{s + 4} + \frac{8/5}{s - 1} \right\} \\ &= -\frac{15}{16} - \frac{t}{4} + \frac{27}{80}e^{-4t} + \frac{8}{5}e^t. \end{aligned}$$

2. Assuming that the solution satisfies the conditions of Corollary ‘Corollary’, we take Laplace transforms of both sides of the differential equation and use the initial conditions,

$$[s^2Y - s(1) - 2] + 2[sY - 1] - Y = \frac{1}{s - 1}.$$

We solve this for the transform $Y(s)$,

$$Y(s) = \frac{1}{(s - 1)(s^2 + 2s - 1)} + \frac{s + 4}{s^2 + 2s - 1}.$$

The inverse transform of this function is the solution of the initial-value problem

$$\begin{aligned} y(t) &= \mathcal{L}^{-1} \left\{ \frac{1}{(s - 1)(s^2 + 2s - 1)} + \frac{s + 4}{s^2 + 2s - 1} \right\} = \mathcal{L}^{-1} \left\{ \frac{1/2}{s - 1} + \frac{s/2 + 5/2}{s^2 + 2s - 1} \right\} \\ &= \frac{1}{2} \left[e^t + \mathcal{L}^{-1} \left\{ \frac{s + 5}{s^2 + 2s - 1} \right\} \right] = \frac{1}{2} \left[e^t + \mathcal{L}^{-1} \left\{ \frac{(s + 1) + 4}{(s + 1)^2 - 2} \right\} \right] \\ &= \frac{1}{2} \left[e^t + e^{-t} \mathcal{L}^{-1} \left\{ \frac{s + 4}{s^2 - 2} \right\} \right] = \frac{1}{2} e^t + \frac{1}{2} e^{-t} \mathcal{L}^{-1} \left\{ \frac{-\sqrt{2} + 1/2}{s + \sqrt{2}} + \frac{\sqrt{2} + 1/2}{s - \sqrt{2}} \right\} \\ &= \frac{1}{2} e^t + \frac{1}{2} e^{-t} \left[\left(\frac{1}{2} - \sqrt{2} \right) e^{-\sqrt{2}t} + \left(\frac{1}{2} + \sqrt{2} \right) e^{\sqrt{2}t} \right] \\ &= \frac{1}{2} e^t + \left(\frac{1}{4} - \frac{\sqrt{2}}{2} \right) e^{-(1+\sqrt{2})t} + \left(\frac{1}{4} + \frac{\sqrt{2}}{2} \right) e^{(-1+\sqrt{2})t}. \end{aligned}$$

3. Assuming that the solution satisfies the conditions of Corollary ‘Corollary’, we take Laplace transforms of both sides of the differential equation and use the initial conditions,

$$[s^2Y] + Y = \frac{2}{s + 1}.$$

We solve this for the transform $Y(s)$,

$$Y(s) = \frac{2}{(s + 1)(s^2 + 1)}.$$

The inverse transform of this function is the solution of the initial-value problem

$$y(t) = \mathcal{L}^{-1} \left\{ \frac{2}{(s + 1)(s^2 + 1)} \right\} = \mathcal{L}^{-1} \left\{ \frac{1}{s + 1} + \frac{1 - s}{s^2 + 1} \right\} = e^{-t} - \cos t + \sin t.$$

4. Assuming that the solution satisfies the conditions of Corollary ‘Corollary’, we take Laplace transforms of both sides of the differential equation and use the initial conditions,

$$[s^2Y - s(0) - 1] + 2[sY] + Y = \frac{1}{s^2}.$$

We solve this for the transform $Y(s)$,

$$Y(s) = \frac{1}{s^2 + 2s + 1} + \frac{1}{s^2(s^2 + 2s + 1)} = \frac{1}{(s + 1)^2} + \frac{1}{s^2(s + 1)^2}.$$

The inverse transform of this function is the solution of the initial-value problem

$$\begin{aligned} y(t) &= \mathcal{L}^{-1} \left\{ \frac{1}{(s + 1)^2} + \frac{1}{s^2(s + 1)^2} \right\} = \mathcal{L}^{-1} \left\{ \frac{2}{(s + 1)^2} - \frac{2}{s} + \frac{1}{s^2} + \frac{2}{s + 1} \right\} \\ &= -2 + t + 2e^{-t} + e^{-t} \mathcal{L}^{-1} \left\{ \frac{2}{s^2} \right\} = t - 2 + 2e^{-t} + 2te^{-t}. \end{aligned}$$

5. Assuming that the solution satisfies the conditions of Corollary ‘Corollary’, we take Laplace transforms of both sides of the differential equation and use the initial conditions,

$$[s^2Y - s(1)] - 2[sY - 1] + Y = \frac{2}{(s - 1)^3}.$$

We solve this for the transform $Y(s)$,

$$Y(s) = \frac{s - 2}{s^2 - 2s + 1} + \frac{2}{(s - 1)^3(s^2 - 2s + 1)} = \frac{s - 2}{(s - 1)^2} + \frac{2}{(s - 1)^5}.$$

The inverse transform of this function is the solution of the initial-value problem

$$\begin{aligned} y(t) &= \mathcal{L}^{-1} \left\{ \frac{s - 2}{(s - 1)^2} + \frac{2}{(s - 1)^5} \right\} = \mathcal{L}^{-1} \left\{ \frac{(s - 1) - 1}{(s - 1)^2} + \frac{2}{(s - 1)^5} \right\} \\ &= \mathcal{L}^{-1} \left\{ \frac{1}{s - 1} - \frac{1}{(s - 1)^2} + \frac{2}{(s - 1)^5} \right\} = e^t \mathcal{L}^{-1} \left\{ \frac{1}{s} - \frac{1}{s^2} + \frac{2}{s^5} \right\} \\ &= e^t \left(1 - t + \frac{2t^4}{4!} \right) = e^t \left(1 - t + \frac{t^4}{12} \right). \end{aligned}$$

6. Assuming that the solution satisfies the conditions of Corollary ‘Corollary’, we take Laplace transforms of both sides of the differential equation and use the initial conditions,

$$[s^2Y - s(1) + 2] + Y = \frac{1}{s^2}.$$

We solve this for the transform $Y(s)$,

$$Y(s) = \frac{s - 2}{s^2 + 1} + \frac{1}{s^2(s^2 + 1)}.$$

The inverse transform of this function is the solution of the initial-value problem

$$y(t) = \mathcal{L}^{-1} \left\{ \frac{s - 2}{s^2 + 1} + \frac{1}{s^2(s^2 + 1)} \right\} = \mathcal{L}^{-1} \left\{ \frac{1}{s^2} + \frac{s - 3}{s^2 + 1} \right\} = t + \cos t - 3 \sin t.$$

7. Assuming that the solution satisfies the conditions of Corollary ‘Corollary’, we take Laplace transforms of both sides of the differential equation and use the initial conditions,

$$[s^2Y - 1] + 2[sY] + 5Y = \frac{1}{(s + 1)^2 + 1}.$$

We solve this for the transform $Y(s)$,

$$Y(s) = \frac{1}{s^2 + 2s + 5} + \frac{1}{(s^2 + 2s + 5)(s^2 + 2s + 2)}.$$

The inverse transform of this function is the solution of the initial-value problem

$$\begin{aligned} y(t) &= \mathcal{L}^{-1} \left\{ \frac{1}{s^2 + 2s + 5} + \frac{1}{(s^2 + 2s + 5)(s^2 + 2s + 2)} \right\} = \mathcal{L}^{-1} \left\{ \frac{2/3}{s^2 + 2s + 5} + \frac{1/3}{s^2 + 2s + 2} \right\} \\ &= \frac{2}{3} \mathcal{L}^{-1} \left\{ \frac{1}{(s+1)^2 + 4} \right\} + \frac{1}{3} \mathcal{L}^{-1} \left\{ \frac{1}{(s+1)^2 + 1} \right\} = \frac{2}{3} e^{-t} \mathcal{L}^{-1} \left\{ \frac{1}{s^2 + 4} \right\} + \frac{1}{3} e^{-t} \mathcal{L}^{-1} \left\{ \frac{1}{s^2 + 1} \right\} \\ &= \frac{1}{3} e^{-t} \sin 2t + \frac{1}{3} e^{-t} \sin t. \end{aligned}$$

8. Assuming that the solution satisfies the conditions of Corollary ‘Corollary’, we take Laplace transforms of both sides of the differential equation and use the initial conditions,

$$[s^2 Y - 2s - 1] + 6[sY - 2] + Y = \frac{3}{s^2 + 9}.$$

We solve this for the transform $Y(s)$,

$$Y(s) = \frac{2s + 13}{s^2 + 6s + 1} + \frac{3}{(s^2 + 9)(s^2 + 6s + 1)}.$$

The inverse transform of this function is the solution of the initial-value problem

$$\begin{aligned} y(t) &= \mathcal{L}^{-1} \left\{ \frac{2s + 13}{s^2 + 6s + 1} + \frac{3}{(s^2 + 9)(s^2 + 6s + 1)} \right\} \\ &= \mathcal{L}^{-1} \left\{ -\frac{9s/194 + 12/194}{s^2 + 9} + \frac{397s/194 + 2588/194}{s^2 + 6s + 1} \right\} \\ &= -\frac{9}{194} \cos 3t - \frac{2}{97} \sin 3t + \frac{1}{194} \mathcal{L}^{-1} \left\{ \frac{397(s+3) + 1397}{(s+3)^2 - 8} \right\} \\ &= -\frac{9}{194} \cos 3t - \frac{2}{97} \sin 3t + \frac{e^{-3t}}{194} \mathcal{L}^{-1} \left\{ \frac{397s + 1397}{s^2 - 8} \right\} \\ &= -\frac{9}{194} \cos 3t - \frac{2}{97} \sin 3t + \frac{e^{-3t}}{194} \mathcal{L}^{-1} \left\{ \frac{(-1397 + 794\sqrt{2})/(4\sqrt{2})}{s + 2\sqrt{2}} + \frac{(794\sqrt{2} + 1397)/(4\sqrt{2})}{s - 2\sqrt{2}} \right\} \\ &= -\frac{9}{194} \cos 3t - \frac{2}{97} \sin 3t + \frac{e^{-3t}}{776\sqrt{2}} \left[(-1397 + 794\sqrt{2})e^{-2\sqrt{2}t} + (1397 + 794\sqrt{2})e^{2\sqrt{2}t} \right] \\ &= -\frac{9}{194} \cos 3t - \frac{2}{97} \sin 3t + \frac{1}{776\sqrt{2}} \left[(-1397 + 794\sqrt{2})e^{-(3+2\sqrt{2})t} + (1397 + 794\sqrt{2})e^{(-3+2\sqrt{2})t} \right]. \end{aligned}$$

9. Assuming that the solution satisfies the conditions of Corollary ‘Corollary’, we take Laplace transforms of both sides of the differential equation and use the initial conditions,

$$[s^2 Y - s(1) + 2] + [sY - 1] - 6Y = \frac{1}{s^2} + \frac{s}{s^2 + 1}.$$

We solve this for the transform $Y(s)$,

$$Y(s) = \frac{s-1}{s^2 + s - 6} + \frac{1}{s^2(s^2 + s - 6)} + \frac{s}{(s^2 + 1)(s^2 + s - 6)}.$$

The inverse transform of this function is the solution of the initial-value problem

$$\begin{aligned} y(t) &= \mathcal{L}^{-1} \left\{ \frac{s-1}{s^2 + s - 6} + \frac{1}{s^2(s^2 + s - 6)} + \frac{s}{(s^2 + 1)(s^2 + s - 6)} \right\} \\ &= \mathcal{L}^{-1} \left\{ \frac{s-1}{(s+3)(s-2)} + \frac{1}{s^2(s+3)(s-2)} + \frac{s}{(s^2 + 1)(s+3)(s-2)} \right\} \end{aligned}$$

$$\begin{aligned}
&= \mathcal{L}^{-1} \left\{ \frac{-1/36}{s} - \frac{1/6}{s^2} + \frac{377/450}{s+3} + \frac{33/100}{s-2} + \frac{-7s/50 + 1/50}{s^2+1} \right\} \\
&= -\frac{1}{36} - \frac{t}{6} + \frac{377}{450}e^{-3t} + \frac{33}{100}e^{2t} - \frac{7}{50}\cos t + \frac{1}{50}\sin t.
\end{aligned}$$

10. Assuming that the solution satisfies the conditions of Corollary ‘Corollary’, we take Laplace transforms of both sides of the differential equation and use the initial conditions,

$$[s^2Y - s(-1) - 2] - 4[sY - (-1)] + 5Y = \frac{1}{(s+3)^2}.$$

We solve this for the transform $Y(s)$,

$$Y(s) = \frac{-s+6}{s^2-4s+5} + \frac{1}{(s+3)^2(s^2-4s+5)}.$$

The inverse transform of this function is the solution of the initial-value problem

$$\begin{aligned}
y(t) &= \mathcal{L}^{-1} \left\{ \frac{-s+6}{s^2-4s+5} + \frac{1}{(s+3)^2(s^2-4s+5)} \right\} \\
&= \mathcal{L}^{-1} \left\{ \frac{5/338}{s+3} + \frac{1/26}{(s+3)^2} + \frac{-343s/338 + 2050/338}{s^2-4s+5} \right\} \\
&= \frac{5}{338}e^{-3t} + \frac{t}{26}e^{-3t} + \frac{1}{338}\mathcal{L}^{-1} \left\{ \frac{-343(s-2) + 1364}{(s-2)^2+1} \right\} \\
&= \frac{5}{338}e^{-3t} + \frac{t}{26}e^{-3t} + \frac{e^{2t}}{338}\mathcal{L}^{-1} \left\{ \frac{-343s + 1364}{s^2+1} \right\} \\
&= \frac{5}{338}e^{-3t} + \frac{t}{26}e^{-3t} + \frac{e^{2t}}{338}(-343\cos t + 1364\sin t).
\end{aligned}$$

11. Assuming that the solution satisfies the conditions of Corollary ‘Corollary’, we take Laplace transforms of both sides of the differential equation and use the initial conditions,

$$[s^2Y - 1] + 4Y = \mathcal{L}\{h(t) - h(t-1)\} = \frac{1}{s} - \frac{e^{-s}}{s}.$$

We solve this for the transform $Y(s)$,

$$Y(s) = \frac{1}{s^2+4} + \frac{1-e^{-s}}{s(s^2+4)}.$$

The inverse transform of this function is the solution of the initial-value problem

$$\begin{aligned}
y(t) &= \mathcal{L}^{-1} \left\{ \frac{1}{s^2+4} + \frac{1-e^{-s}}{s(s^2+4)} \right\} = \mathcal{L}^{-1} \left\{ \frac{1}{s^2+4} + (1-e^{-s}) \left(\frac{1/4}{s} - \frac{s/4}{s^2+4} \right) \right\} \\
&= \frac{1}{2}\sin 2t + \frac{1}{4} - \frac{1}{4}\cos 2t - \left[\frac{1}{4} - \frac{1}{4}\cos 2(t-1) \right] h(t-1).
\end{aligned}$$

12. Assuming that the solution satisfies the conditions of Corollary ‘Corollary’, we take Laplace transforms of both sides of the differential equation and use the initial conditions,

$$[s^2Y] + 2[sY] - 4Y = \mathcal{L}\{\cos^2 t\} = \mathcal{L} \left\{ \frac{1+\cos 2t}{2} \right\} = \frac{1}{2s} + \frac{s}{2(s^2+4)}.$$

We solve this for the transform $Y(s)$,

$$Y(s) = \frac{1}{2s(s^2+2s-4)} + \frac{s}{2(s^2+2s-4)(s^2+4)}.$$

The inverse transform of this function is the solution of the initial-value problem

$$\begin{aligned}
y(t) &= \mathcal{L}^{-1} \left\{ \frac{1}{2s(s^2 + 2s - 4)} + \frac{s}{2(s^2 + 2s - 4)(s^2 + 4)} \right\} \\
&= \mathcal{L}^{-1} \left\{ \frac{-1/8}{s} + \frac{-s/20 + 1/20}{s^2 + 4} + \frac{7s/40 + 12/40}{s^2 + 2s - 4} \right\} \\
&= -\frac{1}{8} - \frac{1}{20} \cos 2t + \frac{1}{40} \sin 2t + \frac{1}{40} \mathcal{L}^{-1} \left\{ \frac{7(s+1) + 5}{(s+1)^2 - 5} \right\} \\
&= -\frac{1}{8} - \frac{1}{20} \cos 2t + \frac{1}{40} \sin 2t + \frac{e^{-t}}{40} \mathcal{L}^{-1} \left\{ \frac{7s + 5}{s^2 - 5} \right\} \\
&= -\frac{1}{8} - \frac{1}{20} \cos 2t + \frac{1}{40} \sin 2t + \frac{e^{-t}}{40} \mathcal{L}^{-1} \left\{ \frac{(7\sqrt{5} - 5)/(2\sqrt{5})}{s + \sqrt{5}} + \frac{(7\sqrt{5} + 5)/(2\sqrt{5})}{s - \sqrt{5}} \right\} \\
&= -\frac{1}{8} - \frac{1}{20} \cos 2t + \frac{1}{40} \sin 2t + \frac{e^{-t}}{80\sqrt{5}} [(7\sqrt{5} - 5)e^{-\sqrt{5}t} + (7\sqrt{5} + 5)e^{\sqrt{5}t}] \\
&= -\frac{1}{8} - \frac{1}{20} \cos 2t + \frac{1}{40} \sin 2t + \frac{1}{80} [(7 - \sqrt{5})e^{-(1+\sqrt{5})t} + (7 + \sqrt{5})e^{(-1+\sqrt{5})t}].
\end{aligned}$$

13. Assuming that the solution satisfies the conditions of Corollary ‘Corollary’, we take Laplace transforms of both sides of the differential equation and use the initial conditions,

$$[s^2Y - 2] - 3[sY] + 2Y = \frac{16}{s^3} + \frac{12}{s+1}.$$

We solve this for the transform $Y(s)$,

$$Y(s) = \frac{2}{s^2 - 3s + 2} + \frac{16}{s^3(s^2 - 3s + 2)} + \frac{12}{(s+1)(s^2 - 3s + 2)}.$$

The inverse transform of this function is the solution of the initial-value problem

$$\begin{aligned}
y(t) &= \mathcal{L}^{-1} \left\{ \frac{2}{s^2 - 3s + 2} + \frac{16}{s^3(s^2 - 3s + 2)} + \frac{12}{(s+1)(s^2 - 3s + 2)} \right\} \\
&= \mathcal{L}^{-1} \left\{ \frac{2}{(s-2)(s-1)} + \frac{16}{s^3(s-2)(s-1)} + \frac{12}{(s+1)(s-2)(s-1)} \right\} \\
&= \mathcal{L}^{-1} \left\{ \frac{14}{s} + \frac{12}{s^2} + \frac{8}{s^3} + \frac{2}{s+1} + \frac{8}{s-2} - \frac{24}{s-1} \right\} \\
&= 14 + 12t + 4t^2 + 2e^{-t} + 8e^{2t} - 24e^t.
\end{aligned}$$

14. Assuming that the solution satisfies the conditions of Corollary ‘Corollary’, we take Laplace transforms of both sides of the differential equation and use the initial conditions,

$$[s^2Y] + 4[sY] - 2Y = \frac{4}{s^2 + 16}.$$

We solve this for the transform $Y(s)$,

$$Y(s) = \frac{4}{(s^2 + 16)(s^2 + 4s - 2)}.$$

The inverse transform of this function is the solution of the initial-value problem

$$\begin{aligned}
y(t) &= \mathcal{L}^{-1} \left\{ \frac{4}{(s^2 + 16)(s^2 + 4s - 2)} \right\} = \mathcal{L}^{-1} \left\{ \frac{-4s/145 - 18/145}{s^2 + 16} + \frac{4s/145 + 34/145}{s^2 + 4s - 2} \right\} \\
&= \frac{1}{145} \left[-4 \cos 4t - \frac{9}{2} \sin 4t + \mathcal{L}^{-1} \left\{ \frac{4(s+2) + 26}{(s+2)^2 - 6} \right\} \right]
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{290} \left[-8 \cos 4t - 9 \sin 4t + 2e^{-2t} \mathcal{L}^{-1} \left\{ \frac{4s + 26}{s^2 - 6} \right\} \right] \\
&= \frac{1}{290} \left[-8 \cos 4t - 9 \sin 4t + 4e^{-2t} \mathcal{L}^{-1} \left\{ \frac{(2\sqrt{6} - 13)/(2\sqrt{6})}{s + \sqrt{6}} + \frac{(2\sqrt{6} + 13)/(2\sqrt{6})}{s - \sqrt{6}} \right\} \right] \\
&= \frac{1}{290} \left\{ -8 \cos 4t - 9 \sin 4t + 4e^{-2t} \left[\left(1 - \frac{13}{2\sqrt{6}}\right) e^{-\sqrt{6}t} + \left(1 + \frac{13}{2\sqrt{6}}\right) e^{\sqrt{6}t} \right] \right\} \\
&= -\frac{1}{290} (8 \cos 4t + 9 \sin 4t) + \frac{\sqrt{6}}{870} \left[(2\sqrt{6} - 13)e^{-(2+\sqrt{6})t} + (2\sqrt{6} + 13)e^{(-2+\sqrt{6})t} \right].
\end{aligned}$$

15. Assuming that the solution satisfies the conditions of Corollary ‘Corollary’, we take Laplace transforms of both sides of the differential equation and use the initial conditions,

$$[s^2Y - 1] + 8[sY] + 41Y = \frac{1}{(s + 2)^2 + 1}.$$

We solve this for the transform $Y(s)$,

$$Y(s) = \frac{1}{s^2 + 8s + 41} + \frac{1}{(s^2 + 4s + 5)(s^2 + 8s + 41)}.$$

The inverse transform of this function is the solution of the initial-value problem

$$\begin{aligned}
y(t) &= \mathcal{L}^{-1} \left\{ \frac{1}{s^2 + 8s + 41} + \frac{1}{(s^2 + 4s + 5)(s^2 + 8s + 41)} \right\} \\
&= \mathcal{L}^{-1} \left\{ \frac{-s/200 + 5/200}{s^2 + 4s + 5} + \frac{s/200 + 199/200}{s^2 + 8s + 41} \right\} \\
&= \frac{1}{200} \mathcal{L}^{-1} \left\{ \frac{-(s + 2) + 7}{(s + 2)^2 + 1} + \frac{(s + 4) + 195}{(s + 4)^2 + 25} \right\} \\
&= \frac{1}{200} \left[e^{-2t} \mathcal{L}^{-1} \left\{ \frac{-s + 7}{s^2 + 1} \right\} + e^{-4t} \mathcal{L}^{-1} \left\{ \frac{s + 195}{s^2 + 25} \right\} \right] \\
&= \frac{1}{200} [e^{-2t}(-\cos t + 7 \sin t) + e^{-4t}(\cos 5t + 39 \sin 5t)].
\end{aligned}$$

16. Assuming that the solution satisfies the conditions of Corollary ‘Corollary’, we take Laplace transforms of both sides of the differential equation and use the initial conditions,

$$[s^2Y] + 2[sY] + Y = \mathcal{L}\{t[h(t) - h(t - 1)]\} = \frac{1}{s^2} - e^{-s} \mathcal{L}\{t + 1\} = \frac{1}{s^2} - e^{-s} \left(\frac{1}{s^2} + \frac{1}{s} \right).$$

We solve this for the transform $Y(s)$,

$$Y(s) = \frac{1}{s^2(s^2 + 2s + 1)} - \frac{e^{-s}(s + 1)}{s^2(s^2 + 2s + 1)} = \frac{1}{s^2(s + 1)^2} - \frac{e^{-s}}{s^2(s + 1)}.$$

The inverse transform of this function is the solution of the initial-value problem

$$\begin{aligned}
y(t) &= \mathcal{L}^{-1} \left\{ \frac{1}{s^2(s + 1)^2} - \frac{e^{-s}}{s^2(s + 1)} \right\} \\
&= \mathcal{L}^{-1} \left\{ -\frac{2}{s} + \frac{1}{s^2} + \frac{2}{s + 1} + \frac{1}{(s + 1)^2} - e^{-s} \left(-\frac{1}{s} + \frac{1}{s^2} + \frac{1}{s + 1} \right) \right\} \\
&= -2 + t + 2e^{-t} + te^{-t} + [1 - (t - 1) - e^{-(t-1)}]h(t - 1) \\
&= -2 + t + 2e^{-t} + te^{-t} + [2 - t - e^{-(t-1)}]h(t - 1).
\end{aligned}$$

17. We set $y'(0) = A$. Assuming that the solution satisfies the conditions of Corollary ‘Corollary’, we take Laplace transforms of both sides of the differential equation and use the initial conditions,

$$[s^2Y - s(1) - A] + 9Y = \frac{s}{s^2 + 4}.$$

We solve this for the transform $Y(s)$,

$$Y(s) = \frac{s + A}{s^2 + 9} + \frac{s}{(s^2 + 4)(s^2 + 9)}.$$

The inverse transform of this function is the solution of the boundary-value problem

$$\begin{aligned} y(t) &= \mathcal{L}^{-1} \left\{ \frac{s + A}{s^2 + 9} + \frac{s}{(s^2 + 4)(s^2 + 9)} \right\} = \mathcal{L}^{-1} \left\{ \frac{s/5}{s^2 + 4} + \frac{4s/5 + A}{s^2 + 9} \right\} \\ &= \frac{1}{5} \cos 2t + \frac{4}{5} \cos 3t + \frac{A}{3} \sin 3t. \end{aligned}$$

Since $y(\pi/2) = -1$,

$$-1 = -\frac{1}{5} - \frac{A}{3} \implies A = \frac{12}{5}.$$

Thus, $y(t) = \frac{1}{5} \cos 2t + \frac{4}{5} \cos 3t + \frac{4}{5} \sin 3t$.

18. We set $y'(0) = A$. Assuming that the solution satisfies the conditions of Corollary 'Corollary', we take Laplace transforms of both sides of the differential equation and use the initial conditions,

$$[s^2Y - s(1) - A] + 3[sY - 1] - 4Y = \frac{2}{s + 4}.$$

We solve this for the transform $Y(s)$,

$$Y(s) = \frac{s + A + 3}{s^2 + 3s - 4} + \frac{2}{(s + 4)(s^2 + 3s - 4)}.$$

The inverse transform of this function is the solution of the boundary-value problem

$$\begin{aligned} y(t) &= \mathcal{L}^{-1} \left\{ \frac{s + A + 3}{s^2 + 3s - 4} + \frac{2}{(s + 4)(s^2 + 3s - 4)} \right\} = \mathcal{L}^{-1} \left\{ \frac{s + A + 3}{(s + 4)(s - 1)} + \frac{2}{(s + 4)^2(s - 1)} \right\} \\ &= \mathcal{L}^{-1} \left\{ \frac{5A/25 + 22/25}{s - 1} + \frac{(3 - 5A)/25}{s + 4} - \frac{2/5}{(s + 4)^2} \right\} \\ &= \left(\frac{A}{5} + \frac{22}{25} \right) e^t + \left(\frac{3}{25} - \frac{A}{5} \right) e^{-4t} - \frac{2t}{5} e^{-4t}. \end{aligned}$$

Since $y(1) = 1$,

$$1 = \left(\frac{A}{5} + \frac{22}{25} \right) e + \left(\frac{3}{25} - \frac{A}{5} \right) e^{-4} - \frac{2}{5} e^{-4} \implies A = \frac{25e^4 - 22e^5 + 7}{5(e^5 - 1)}.$$

Thus,

$$\begin{aligned} y(t) &= \left[\frac{25e^4 - 22e^5 + 7}{25(e^5 - 1)} + \frac{22}{25} \right] e^t + \left[\frac{3}{25} - \frac{25e^4 - 22e^5 + 7}{25(e^5 - 1)} \right] e^{-4t} - \frac{2t}{5} e^{-4t} \\ &= \left(\frac{5e^4 - 3}{5e^5 - 5} \right) e^t + \left(\frac{5e^5 - 5e^4 - 2}{5e^5 - 5} \right) e^{-4t} - \frac{2t}{5} e^{-4t}. \end{aligned}$$

19. We set $y'(0) = A$. Assuming that the solution satisfies the conditions of Corollary 'Corollary', we take Laplace transforms of both sides of the differential equation and use the initial conditions,

$$[s^2Y - A] + 2[sY] + 5Y = \frac{1}{(s + 1)^2 + 1}.$$

We solve this for the transform $Y(s)$,

$$Y(s) = \frac{A}{s^2 + 2s + 5} + \frac{1}{(s^2 + 2s + 5)(s^2 + 2s + 2)}.$$

The inverse transform of this function is the solution of the boundary-value problem

$$\begin{aligned} y(t) &= \mathcal{L}^{-1} \left\{ \frac{A}{s^2 + 2s + 5} + \frac{1}{(s^2 + 2s + 5)(s^2 + 2s + 2)} \right\} = \mathcal{L}^{-1} \left\{ \frac{1/3}{s^2 + 2s + 2} + \frac{A - 1/3}{s^2 + 2s + 5} \right\} \\ &= \frac{1}{3} \mathcal{L}^{-1} \left\{ \frac{1}{(s+1)^2 + 1} \right\} + \left(A - \frac{1}{3} \right) \mathcal{L}^{-1} \left\{ \frac{1}{(s+1)^2 + 4} \right\} \\ &= \frac{1}{3} e^{-t} \mathcal{L}^{-1} \left\{ \frac{1}{s^2 + 1} \right\} + \left(A - \frac{1}{3} \right) e^{-t} \mathcal{L}^{-1} \left\{ \frac{1}{s^2 + 4} \right\} \\ &= \frac{1}{3} e^{-t} \sin t + \frac{1}{2} \left(A - \frac{1}{3} \right) e^{-t} \sin 2t. \end{aligned}$$

Since $y(\pi/4) = 1$,

$$1 = \frac{1}{3} e^{-\pi/4} \left(\frac{1}{\sqrt{2}} \right) + \frac{1}{2} \left(A - \frac{1}{3} \right) e^{-\pi/4} \implies A = \frac{1}{3} - \frac{\sqrt{2}}{3} + 2e^{\pi/4}.$$

Thus, $y(t) = \frac{1}{3} e^{-t} \sin t + \left(e^{\pi/4} - \frac{1}{3\sqrt{2}} \right) e^{-t} \sin 2t$.

20. (a) If we let $y(0) = A$ and $y'(0) = B$, and take Laplace transforms of the differential equation,

$$(s^2 Y - As - B) + 2(sY - A) - 3Y = \frac{2}{s^2 + 4} \implies Y(s) = \frac{As + (B + 2A) + \frac{2}{s^2 + 4}}{s^2 + 2s - 3}.$$

Since the partial fraction decomposition of $\frac{2}{(s^2 + 4)(s^2 + 2s - 3)}$ is

$$\frac{2}{(s^2 + 4)(s^2 + 2s - 3)} = \frac{1/10}{s - 1} - \frac{1/26}{s + 3} - \frac{4s/65 + 14/65}{s^2 + 4},$$

we can write that

$$Y(s) = -\frac{4s/65 + 14/65}{s^2 + 4} + \frac{C}{s - 1} + \frac{D}{s + 3}.$$

Inverse transforms give

$$y(t) = -\frac{7}{65} \sin 2t - \frac{4}{65} \cos 2t + Ce^t + De^{-3t}.$$

The initial conditions require

$$3 = \frac{4}{65} + Ce^{\pi/2} + De^{-3\pi/2}, \quad -1 = \frac{14}{65} + Ce^{\pi/2} - 3De^{-3\pi/2}.$$

The solution of these is $C = (19/10)e^{-\pi/2}$ and $D = (27/26)e^{3\pi/2}$. Thus,

$$y(t) = -\frac{7}{65} \sin 2t - \frac{4}{65} \cos 2t + \frac{19}{10} e^{t-\pi/2} + \frac{27}{26} e^{-3(t-\pi/2)}.$$

- (b) If we set $u = t - \pi/2$, the initial-value problem for $y(u)$ is

$$y'' + 2y' - 3y = \sin 2(u + \pi/2) = -\sin 2u, \quad y(0) = 3, \quad y'(0) = -1.$$

When we take Laplace transforms,

$$(s^2 Y - 3s + 1) + 2(sY - 3) - 3Y = \frac{-2}{s^2 + 4} \implies Y(s) = \frac{3s + 5 - \frac{2}{s^2 + 4}}{s^2 + 2s - 3}.$$

Partial fractions give

$$Y(s) = \frac{19/10}{s-1} + \frac{27/26}{s+3} + \frac{14s/65 + 4/65}{s^2 + 4},$$

and therefore

$$y(u) = \frac{19}{10}e^u + \frac{27}{26}e^{-3u} + \frac{7}{65}\sin 2u + \frac{4}{65}\cos 2u.$$

Returning now to variable t ,

$$\begin{aligned} y(t) &= \frac{19}{10}e^{t-\pi/2} + \frac{27}{26}e^{-3(t-\pi/2)} + \frac{7}{65}\sin 2(t-\pi/2) + \frac{4}{65}\cos 2(t-\pi/2) \\ &= \frac{19}{10}e^{t-\pi/2} + \frac{27}{26}e^{-3(t-\pi/2)} - \frac{7}{65}\sin 2t - \frac{4}{65}\cos 2t. \end{aligned}$$

21. Assuming that the solution satisfies the conditions of Corollary ‘Corollary’, we take Laplace transforms of both sides of the differential equation and use the initial conditions,

$$[s^2Y - s(1)] - 4[sY - 1] + 3Y = F(s).$$

We solve this for the transform $Y(s)$,

$$Y(s) = \frac{s-4}{s^2-4s+3} + \frac{F(s)}{s^2-4s+3} = \frac{s-4}{(s-1)(s-3)} + \frac{F(s)}{(s-1)(s-3)}.$$

The inverse transform of this function is the solution of the initial-value problem

$$\begin{aligned} y(t) &= \mathcal{L}^{-1} \left\{ \frac{s-4}{(s-1)(s-3)} + \frac{F(s)}{(s-1)(s-3)} \right\} = \mathcal{L}^{-1} \left\{ \frac{3/2}{s-1} - \frac{1/2}{s-3} + \left(\frac{-1/2}{s-1} + \frac{1/2}{s-3} \right) F(s) \right\} \\ &= \frac{3}{2}e^t - \frac{1}{2}e^{3t} + \frac{1}{2} \int_0^t [-e^{t-u} + e^{3(t-u)}]f(u) du. \end{aligned}$$

22. Assuming that the solution satisfies the conditions of Corollary ‘Corollary’, we take Laplace transforms of both sides of the differential equation and use the initial conditions,

$$[s^2Y] + 4[sY] + 6Y = F(s).$$

We solve this for the transform $Y(s)$,

$$Y(s) = \frac{F(s)}{s^2 + 4s + 6}.$$

$$\text{Since } \mathcal{L}^{-1} \left\{ \frac{1}{s^2 + 4s + 6} \right\} = \mathcal{L}^{-1} \left\{ \frac{1}{(s+2)^2 + 2} \right\} = e^{-2t} \mathcal{L}^{-1} \left\{ \frac{1}{s^2 + 2} \right\} = \frac{e^{-2t}}{\sqrt{2}} \sin \sqrt{2}t,$$

$$y(t) = \mathcal{L}^{-1} \left\{ \frac{F(s)}{s^2 + 4s + 6} \right\} = \frac{1}{\sqrt{2}} \int_0^t e^{-2(t-u)} \sin \sqrt{2}(t-u) f(u) du.$$

23. We set $y(0) = A$ and $y'(0) = B$. Assuming that the solution satisfies the conditions of Corollary ‘Corollary’, we take Laplace transforms of both sides of the differential equation and use the initial conditions,

$$[s^2Y - As - B] + 16Y = F(s).$$

We solve this for the transform $Y(s)$,

$$Y(s) = \frac{As + B}{s^2 + 16} + \frac{F(s)}{s^2 + 16}.$$

The inverse transform of this function is the solution of the initial-value problem

$$\begin{aligned} y(t) &= A \cos 4t + \frac{B}{4} \sin 4t + \frac{1}{4} \int_0^t \sin 4(t-u) f(u) du \\ &= A \cos 4t + C \sin 4t + \frac{1}{4} \int_0^t \sin 4(t-u) f(u) du. \end{aligned}$$

24. We set $y(0) = A$ and $y'(0) = B$. Assuming that the solution satisfies the conditions of Corollary 'Corollary', we take Laplace transforms of both sides of the differential equation and use the initial conditions,

$$[s^2 Y - As - B] + 3[sY - A] + 2Y = F(s-1).$$

We solve this for the transform $Y(s)$,

$$Y(s) = \frac{As + B + 3A}{s^2 + 3s + 2} + \frac{F(s-1)}{s^2 + 3s + 2} = \frac{As + B + 3A}{(s+1)(s+2)} + \frac{F(s-1)}{(s+1)(s+2)}.$$

The inverse transform of this function is the solution of the initial-value problem

$$\begin{aligned} y(t) &= \mathcal{L}^{-1} \left\{ \frac{As + B + 3A}{(s+1)(s+2)} + \frac{F(s-1)}{(s+1)(s+2)} \right\} \\ &= \mathcal{L}^{-1} \left\{ \frac{2A+B}{s+1} - \frac{A+B}{s+2} + \left(\frac{1}{s+1} - \frac{1}{s+2} \right) F(s-1) \right\} \\ &= (2A+B)e^{-t} - (A+B)e^{-2t} + \int_0^t [e^{-(t-u)} - e^{-2(t-u)}] e^u f(u) du \\ &= Ce^{-t} + De^{-2t} + \int_0^t (e^{2u-t} - e^{3u-2t}) f(u) du. \end{aligned}$$

25. Since $\mathcal{L}^{-1} \left\{ \frac{1}{s} \right\} = 1$ and $\mathcal{L}^{-1} \left\{ \frac{1}{s+1} \right\} = e^{-t}$,

$$f(t) = \int_0^t e^{-u} du = \{-e^{-u}\}_0^t = 1 - e^{-t}.$$

26. Since $\mathcal{L}^{-1} \left\{ \frac{1}{s^2+1} \right\} = \sin t$ and $\mathcal{L}^{-1} \left\{ \frac{1}{s^2+4} \right\} = \frac{1}{2} \sin 2t$,
- $$\begin{aligned} f(t) &= \frac{1}{2} \int_0^t \sin(t-u) \sin 2u du = \frac{1}{4} \int_0^t [\cos(t-3u) - \cos(t+u)] du \\ &= \frac{1}{4} \left\{ -\frac{1}{3} \sin(t-3u) - \sin(t+u) \right\}_0^t = -\frac{1}{6} \sin 2t + \frac{1}{3} \sin t. \end{aligned}$$

27. Since $\mathcal{L}^{-1} \left\{ \frac{1}{s+4} \right\} = e^{-4t}$ and

$$\mathcal{L}^{-1} \left\{ \frac{s}{s^2-2} \right\} = \mathcal{L}^{-1} \left\{ \frac{1/2}{s+\sqrt{2}} + \frac{1/2}{s-\sqrt{2}} \right\} = \frac{1}{2} (e^{-\sqrt{2}t} + e^{\sqrt{2}t}),$$

it follows that

$$\begin{aligned} f(t) &= \frac{1}{2} \int_0^t (e^{-\sqrt{2}u} + e^{\sqrt{2}u}) e^{-4(t-u)} du = \frac{1}{2} \int_0^t [e^{-4t+(4-\sqrt{2})u} + e^{-4t+(4+\sqrt{2})u}] du \\ &= \frac{1}{2} \left\{ \frac{e^{-4t+(4-\sqrt{2})u}}{4-\sqrt{2}} + \frac{e^{-4t+(4+\sqrt{2})u}}{4+\sqrt{2}} \right\}_0^t = \frac{1}{2} \left[\frac{e^{-\sqrt{2}t} - e^{-4t}}{4-\sqrt{2}} + \frac{e^{\sqrt{2}t} - e^{-4t}}{4+\sqrt{2}} \right] \\ &= \frac{1}{2} \left[\left(\frac{4+\sqrt{2}}{14} \right) e^{-\sqrt{2}t} + \left(\frac{4-\sqrt{2}}{14} \right) e^{\sqrt{2}t} + \left(-\frac{4+\sqrt{2}}{14} - \frac{4-\sqrt{2}}{14} \right) e^{-4t} \right] \\ &= \left(\frac{4+\sqrt{2}}{28} \right) e^{-\sqrt{2}t} + \left(\frac{4-\sqrt{2}}{28} \right) e^{\sqrt{2}t} - \frac{2}{7} e^{-4t}. \end{aligned}$$

28. Since $\mathcal{L}^{-1}\left\{\frac{s}{s^2-4}\right\} = \mathcal{L}^{-1}\left\{\frac{1/2}{s+2} + \frac{1/2}{s-2}\right\} = \frac{1}{2}(e^{-2t} + e^{2t})$, and $\mathcal{L}^{-1}\left\{\frac{1}{s^2-9}\right\} = \mathcal{L}^{-1}\left\{\frac{-1/6}{s+3} + \frac{1/6}{s-3}\right\} = \frac{1}{6}(e^{3t} - e^{-3t})$, it follows that

$$\begin{aligned} f(t) &= \frac{1}{12} \int_0^t [e^{-2(t-u)} + e^{2(t-u)}][e^{3u} - e^{-3u}] du = \frac{1}{12} \int_0^t (e^{5u-2t} + e^{u+2t} - e^{-u-2t} - e^{2t-5u}) du \\ &= \frac{1}{12} \left\{ \frac{1}{5} e^{5u-2t} + e^{u+2t} + e^{-u-2t} + \frac{1}{5} e^{2t-5u} \right\}_0^t \\ &= \frac{1}{12} \left(\frac{1}{5} e^{3t} + e^{3t} + e^{-3t} + \frac{1}{5} e^{-3t} - \frac{1}{5} e^{-2t} - e^{2t} - e^{-2t} - \frac{1}{5} e^{2t} \right) \\ &= \frac{1}{10} (e^{3t} + e^{-3t} - e^{2t} - e^{-2t}). \end{aligned}$$

29. We set $y(0) = A$ and $y'(0) = B$. Assuming that the solution satisfies the conditions of Corollary 'Corollary', we take Laplace transforms of both sides of the differential equation and use the initial conditions,

$$[s^2Y - As - B] - 2[sY - A] + 4Y = \frac{2}{s^3}.$$

We solve this for the transform $Y(s)$,

$$Y(s) = \frac{As + B - 2A}{s^2 - 2s + 4} + \frac{2}{s^3(s^2 - 2s + 4)}.$$

The inverse transform of this function is the solution of the differential equation

$$\begin{aligned} y(t) &= \mathcal{L}^{-1} \left\{ \frac{As + B - 2A}{s^2 - 2s + 4} + \frac{2}{s^3(s^2 - 2s + 4)} \right\} = \mathcal{L}^{-1} \left\{ \frac{As + B - 2A}{s^2 - 2s + 4} + \frac{1/4}{s^2} + \frac{1/2}{s^3} - \frac{1/4}{s^2 - 2s + 4} \right\} \\ &= \mathcal{L}^{-1} \left\{ \frac{As + C}{(s-1)^2 + 3} + \frac{1/4}{s^2} + \frac{1/2}{s^3} \right\} = \mathcal{L}^{-1} \left\{ \frac{A(s-1) + D}{(s-1)^2 + 3} + \frac{1/4}{s^2} + \frac{1/2}{s^3} \right\} \\ &= e^t \mathcal{L}^{-1} \left\{ \frac{As + D}{s^2 + 3} \right\} + \frac{t}{4} + \frac{t^2}{4} = e^t \left(A \cos \sqrt{3}t + \frac{D}{\sqrt{3}} \sin \sqrt{3}t \right) + \frac{t}{4} + \frac{t^2}{4} \\ &= e^t (A \cos \sqrt{3}t + E \sin \sqrt{3}t) + \frac{t}{4} + \frac{t^2}{4}. \end{aligned}$$

30. We set $y(0) = A$ and $y'(0) = B$. Assuming that the solution satisfies the conditions of Corollary 'Corollary', we take Laplace transforms of both sides of the differential equation and use the initial conditions,

$$[s^2Y - As - B] - 2[sY - A] + Y = \frac{2}{(s-1)^3}.$$

We solve this for the transform $Y(s)$,

$$Y(s) = \frac{As + B - 2A}{s^2 - 2s + 1} + \frac{2}{(s-1)^3(s^2 - 2s + 1)} = \frac{As + B - 2A}{(s-1)^2} + \frac{2}{(s-1)^5}.$$

The inverse transform of this function is the solution of the differential equation

$$\begin{aligned} y(t) &= \mathcal{L}^{-1} \left\{ \frac{As + B - 2A}{(s-1)^2} + \frac{2}{(s-1)^5} \right\} = \mathcal{L}^{-1} \left\{ \frac{A(s-1) + B - A}{(s-1)^2} + \frac{2}{(s-1)^5} \right\} \\ &= \mathcal{L}^{-1} \left\{ \frac{A}{s-1} + \frac{B-A}{(s-1)^2} + \frac{2}{(s-1)^5} \right\} = Ae^t + (B-A)te^t + \frac{2}{4!}t^4e^t \\ &= Ae^t + Cte^t + \frac{t^4}{12}e^t. \end{aligned}$$

31. We set $y(0) = A$ and $y'(0) = B$. Assuming that the solution satisfies the conditions of Corollary 'Corollary', we take Laplace transforms of both sides of the differential equation and use the initial conditions,

$$[s^2Y - As - B] + Y = F(s).$$

We solve this for the transform $Y(s)$,

$$Y(s) = \frac{As + B}{s^2 + 1} + \frac{F(s)}{s^2 + 1}.$$

The inverse transform of this function is the solution of the differential equation

$$y(t) = \mathcal{L}^{-1} \left\{ \frac{As + B}{s^2 + 1} + \frac{F(s)}{s^2 + 1} \right\} = A \cos t + B \sin t + \int_0^t f(u) \sin(t - u) du.$$

32. We set $y(0) = A$ and $y'(0) = B$. Assuming that the solution satisfies the conditions of Corollary 'Corollary', we take Laplace transforms of both sides of the differential equation and use the initial conditions,

$$[s^2Y - As - B] + 2[sY - A] + 5Y = \frac{1}{(s + 1)^2 + 1}.$$

We solve this for the transform $Y(s)$,

$$Y(s) = \frac{As + B + 2A}{s^2 + 2s + 5} + \frac{1}{(s^2 + 2s + 5)(s^2 + 2s + 2)}.$$

The inverse transform of this function is the solution of the differential equation

$$\begin{aligned} y(t) &= \mathcal{L}^{-1} \left\{ \frac{As + B + 2A}{s^2 + 2s + 5} + \frac{1}{(s^2 + 2s + 5)(s^2 + 2s + 2)} \right\} = \mathcal{L}^{-1} \left\{ \frac{1/3}{s^2 + 2s + 2} + \frac{As + B + 2A - 1/3}{s^2 + 2s + 5} \right\} \\ &= \mathcal{L}^{-1} \left\{ \frac{1/3}{(s + 1)^2 + 1} + \frac{A(s + 1) + B + A - 1/3}{(s + 1)^2 + 4} \right\} = \frac{1}{3} e^{-t} \mathcal{L}^{-1} \left\{ \frac{1}{s^2 + 1} \right\} + e^{-t} \mathcal{L}^{-1} \left\{ \frac{As + C}{s^2 + 4} \right\} \\ &= \frac{1}{3} e^{-t} \sin t + e^{-t} \left(A \cos 2t + \frac{C}{2} \sin 2t \right) = \frac{1}{3} e^{-t} \sin t + e^{-t} (A \cos 2t + D \sin 2t). \end{aligned}$$

33. We set $y(0) = A$ and $y'(0) = B$. Assuming that the solution satisfies the conditions of Corollary 'Corollary', we take Laplace transforms of both sides of the differential equation and use the initial conditions,

$$[s^2Y - As - B] + 4[sY - A] + Y = \frac{1}{s^2} + \frac{2}{s}.$$

We solve this for the transform $Y(s)$,

$$Y(s) = \frac{As + B + 4A}{s^2 + 4s + 1} + \frac{2s + 1}{s^2(s^2 + 4s + 1)}.$$

The inverse transform of this function is the solution of the differential equation

$$\begin{aligned} y(t) &= \mathcal{L}^{-1} \left\{ \frac{As + B + 4A}{s^2 + 4s + 1} + \frac{2s + 1}{s^2(s^2 + 4s + 1)} \right\} = \mathcal{L}^{-1} \left\{ \frac{As + B + 4A}{s^2 + 4s + 1} - \frac{2}{s} + \frac{1}{s^2} + \frac{2s + 7}{s^2 + 4s + 1} \right\} \\ &= \mathcal{L}^{-1} \left\{ \frac{Cs + D}{s^2 + 4s + 1} - \frac{2}{s} + \frac{1}{s^2} \right\} = \mathcal{L}^{-1} \left\{ \frac{C(s + 2) + D - 2C}{(s + 2)^2 - 3} \right\} - 2 + t \\ &= t - 2 + e^{-2t} \mathcal{L}^{-1} \left\{ \frac{Cs + E}{s^2 - 3} \right\} = t - 2 + e^{-2t} \mathcal{L}^{-1} \left\{ \frac{F}{s + \sqrt{3}} + \frac{G}{s - \sqrt{3}} \right\} \\ &= t - 2 + e^{-2t} (F e^{-\sqrt{3}t} + G e^{\sqrt{3}t}) = t - 2 + F e^{-(2+\sqrt{3})t} + G e^{(-2+\sqrt{3})t}. \end{aligned}$$

34. We set $y(0) = A$ and $y'(0) = B$. Assuming that the solution satisfies the conditions of Corollary 'Corollary', we take Laplace transforms of both sides of the differential equation and use the initial conditions,

$$[s^2Y - As - B] - 4Y = F(s).$$

We solve this for the transform $Y(s)$,

$$Y(s) = \frac{As + B}{s^2 - 4} + \frac{F(s)}{s^2 - 4}.$$

The inverse transform of this function is the solution of the differential equation

$$\begin{aligned} y(t) &= \mathcal{L}^{-1} \left\{ \frac{As + B}{s^2 - 4} + \frac{F(s)}{s^2 - 4} \right\} = \mathcal{L}^{-1} \left\{ \frac{As + B}{(s + 2)(s - 2)} + \frac{F(s)}{(s + 2)(s - 2)} \right\} \\ &= \mathcal{L}^{-1} \left\{ \frac{C}{s + 2} + \frac{D}{s - 2} + \left(\frac{-1/4}{s + 2} + \frac{1/4}{s - 2} \right) F(s) \right\} \\ &= Ce^{-2t} + De^{2t} + \frac{1}{4} \int_0^t [e^{2(t-u)} - e^{-2(t-u)}] f(u) du. \end{aligned}$$

35. We set $y(0) = A$ and $y'(0) = B$. Assuming that the solution of $y'' + 9y = te^{ti}$ satisfies the conditions of Corollary 'Corollary', we take Laplace transforms of both sides of the differential equation and use the initial conditions,

$$[s^2Y - As - B] + 9Y = \frac{1}{(s - i)^2}.$$

We solve this for the transform $Y(s)$,

$$Y(s) = \frac{As + B}{s^2 + 9} + \frac{1}{(s - i)^2(s^2 + 9)}.$$

The inverse transform of this function is the solution of the differential equation

$$\begin{aligned} y(t) &= \mathcal{L}^{-1} \left\{ \frac{As + B}{s^2 + 9} + \frac{1}{(s - i)^2(s^2 + 9)} \right\} \\ &= \mathcal{L}^{-1} \left\{ \frac{As + B}{s^2 + 9} - \frac{i/32}{s - i} + \frac{1/8}{(s - i)^2} + \frac{-is/32 - 5/32}{s^2 + 9} \right\} \\ &= \mathcal{L}^{-1} \left\{ \frac{Cs + D}{s^2 + 9} - \frac{i/32}{s - i} + \frac{1/8}{(s - i)^2} \right\} = C \cos 3t + \frac{D}{3} \sin 3t - \frac{i}{32} e^{ti} + \frac{t}{8} e^{ti}. \end{aligned}$$

Since C and D can be complex, if we take imaginary parts, we get

$$y(t) = E \cos 3t + F \sin 3t - \frac{1}{32} \cos t + \frac{t}{8} \sin t.$$

36. We set $y(0) = A$ and $y'(0) = B$. Assuming that the solution of $y'' - 2y' + 3y = te^{2ti}$ satisfies the conditions of Corollary 'Corollary', we take Laplace transforms of both sides of the differential equation and use the initial conditions,

$$[s^2Y - As - B] - 2[sY - A] + 3Y = \frac{1}{(s - 2i)^2}.$$

We solve this for the transform $Y(s)$,

$$Y(s) = \frac{As + B - 2A}{s^2 - 2s + 3} + \frac{1}{(s - 2i)^2(s^2 - 2s + 3)}.$$

The inverse transform of this function is the solution of the differential equation

$$\begin{aligned}
y(t) &= \mathcal{L}^{-1} \left\{ \frac{As + B - 2A}{s^2 - 2s + 3} + \frac{1}{(s - 2i)^2(s^2 - 2s + 3)} \right\} \\
&= \mathcal{L}^{-1} \left\{ \frac{As + B - 2A}{s^2 - 2s + 3} + \frac{-62/289 + 44i/289}{s - 2i} + \frac{-1/17 + 4i/17}{(s - 2i)^2} + \frac{Cs + D}{s^2 - 2s + 3} \right\} \\
&= \mathcal{L}^{-1} \left\{ \frac{Es + F}{s^2 - 2s + 3} + \frac{-62/289 + 44i/289}{s - 2i} + \frac{-1/17 + 4i/17}{(s - 2i)^2} \right\} \\
&= \left(-\frac{62}{289} + \frac{44i}{289} \right) e^{2ti} + \left(-\frac{1}{17} + \frac{4i}{17} \right) te^{2ti} + \mathcal{L}^{-1} \left\{ \frac{E(s - 1) + F + E}{(s - 1)^2 + 2} \right\} \\
&= \left(-\frac{62}{289} + \frac{44i}{289} \right) e^{2ti} + \left(-\frac{1}{17} + \frac{4i}{17} \right) te^{2ti} + e^t \mathcal{L}^{-1} \left\{ \frac{Es + G}{s^2 + 2} \right\} \\
&= \left(-\frac{62}{289} + \frac{44i}{289} \right) e^{2ti} + \left(-\frac{1}{17} + \frac{4i}{17} \right) te^{2ti} + e^t \left(E \cos \sqrt{2}t + \frac{G}{\sqrt{2}} \sin \sqrt{2}t \right).
\end{aligned}$$

Constants E and G could be complex, but if we take real parts, we get

$$y(t) = -\frac{62}{289} \cos 2t - \frac{44}{289} \sin 2t - \frac{t}{17} \cos 2t - \frac{4t}{17} \sin 2t + e^t (H \cos \sqrt{2}t + J \sin \sqrt{2}t).$$

- 37.** Assuming that the solution satisfies the conditions of Corollary 'C0rrollary', we take Laplace transforms of both sides of the differential equation and use the initial conditions,

$$[s^3Y - s^2(1) + 2] - 3[s^2Y - s(1)] + 3[sY - 1] - Y = \frac{2}{(s - 1)^3}.$$

We solve this for the transform $Y(s)$,

$$Y(s) = \frac{s^2 - 3s + 1}{s^3 - 3s^2 + 3s - 1} + \frac{2}{(s - 1)^3(s^3 - 3s^2 + 3s - 1)} = \frac{s^2 - 3s + 1}{(s - 1)^3} + \frac{2}{(s - 1)^6}.$$

The inverse transform of this function is the solution of the initial-value problem

$$\begin{aligned}
y(t) &= \mathcal{L}^{-1} \left\{ \frac{s^2 - 3s + 1}{(s - 1)^3} + \frac{2}{(s - 1)^6} \right\} = \mathcal{L}^{-1} \left\{ \frac{1}{s - 1} - \frac{1}{(s - 1)^2} - \frac{1}{(s - 1)^3} + \frac{2}{(s - 1)^6} \right\} \\
&= e^t \mathcal{L}^{-1} \left\{ \frac{1}{s} - \frac{1}{s^2} - \frac{1}{s^3} + \frac{2}{s^6} \right\} = e^t \left(1 - t - \frac{t^2}{2} + \frac{t^5}{60} \right).
\end{aligned}$$

- 38.** We set $y(0) = A$, $y'(0) = B$, and $y''(0) = C$. Assuming that the solution satisfies the conditions of Corollary 'C0rrollary', we take Laplace transforms of both sides of the differential equation and use the initial conditions,

$$[s^3Y - As^2 - Bs - C] - 3[s^2Y - As - B] + 3[sY - A] - Y = \frac{2}{(s - 1)^3}.$$

We solve this for the transform $Y(s)$,

$$\begin{aligned}
Y(s) &= \frac{As^2 + (B - 3A)s + (C - 3B + 3A)}{s^3 - 3s^2 + 3s - 1} + \frac{2}{(s - 1)^3(s^3 - 3s^2 + 3s - 1)} \\
&= \frac{As^2 + (B - 3A)s + (C - 3B + 3A)}{(s - 1)^3} + \frac{2}{(s - 1)^6}.
\end{aligned}$$

The inverse transform of this function is the solution of the differential equation

$$y(t) = \mathcal{L}^{-1} \left\{ \frac{As^2 + (B - 3A)s + (C - 3B + 3A)}{(s - 1)^3} + \frac{2}{(s - 1)^6} \right\}$$

$$\begin{aligned}
&= \mathcal{L}^{-1} \left\{ \frac{D}{s-1} + \frac{E}{(s-1)^2} + \frac{F}{(s-1)^3} + \frac{2}{(s-1)^6} \right\} \\
&= e^t \mathcal{L}^{-1} \left\{ \frac{D}{s} + \frac{E}{s^2} + \frac{F}{s^3} + \frac{2}{s^6} \right\} = e^t \left(D + Et + \frac{Ft^2}{2} + \frac{t^5}{60} \right) \\
&= e^t \left(D + Et + Gt^2 + \frac{t^5}{60} \right).
\end{aligned}$$

39. The initial-value problem is

$$\frac{1}{5} \frac{d^2x}{dt^2} + 10x = 0 \implies x'' + 50x = 0, \quad x(0) = -0.03, \quad x'(0) = 0.$$

Assuming that the solution satisfies the conditions of Corollary ‘Corollary’, we take Laplace transforms of both sides of the differential equation and use the initial conditions,

$$[s^2X + 0.03s] + 50X = 0.$$

We solve this for the transform $X(s)$,

$$X(s) = -\frac{0.03s}{s^2 + 50}.$$

The inverse transform of this function is the solution of the initial-value problem

$$x(t) = \mathcal{L}^{-1} \left\{ -\frac{0.03s}{s^2 + 50} \right\} = -0.03 \cos 5\sqrt{2}t \text{ m.}$$

40. The initial-value problem is

$$2 \frac{d^2x}{dt^2} + 16x = 0, \quad x(0) = 0.1, \quad x'(0) = 0.$$

Assuming that the solution satisfies the conditions of Corollary ‘Corollary’, we take Laplace transforms of both sides of the differential equation and use the initial conditions,

$$2[s^2X - 0.1s] + 16X = 0.$$

We solve this for the transform $X(s)$,

$$X(s) = \frac{0.2s}{2s^2 + 16} = \frac{0.1s}{s^2 + 8}.$$

The inverse transform of this function is the solution of the initial-value problem

$$x(t) = \mathcal{L}^{-1} \left\{ \frac{0.1s}{s^2 + 8} \right\} = 0.1 \cos 2\sqrt{2}t \text{ m.}$$

41. The initial-value problem is

$$\frac{1}{5} \frac{d^2x}{dt^2} + \frac{3}{2} \frac{dx}{dt} + 10x = 0 \implies 2x'' + 15x' + 100x = 0, \quad x(0) = -0.03, \quad x'(0) = 0.$$

Assuming that the solution satisfies the conditions of Corollary ‘Corollary’, we take Laplace transforms of both sides of the differential equation and use the initial conditions,

$$2[s^2X + 0.03s] + 15[sX + 0.03] + 100X = 0.$$

We solve this for the transform $X(s)$,

$$X(s) = -\frac{0.06s + 0.45}{2s^2 + 15s + 100}.$$

The inverse transform of this function is the solution of the initial-value problem

$$\begin{aligned}
x(t) &= \mathcal{L}^{-1} \left\{ -\frac{0.06s + 0.45}{2s^2 + 15s + 100} \right\} = -\mathcal{L}^{-1} \left\{ \frac{0.03s + 0.225}{s^2 + 15s/2 + 50} \right\} \\
&= -\mathcal{L}^{-1} \left\{ \frac{0.03(s + 15/4) + 0.1125}{(s + 15/4)^2 + 575/16} \right\} = -e^{-15t/4} \mathcal{L}^{-1} \left\{ \frac{0.03s + 0.1125}{s^2 + 575/16} \right\} \\
&= -e^{-15t/4} \left[0.03 \cos \frac{5\sqrt{23}t}{4} + 0.1125 \left(\frac{4}{5\sqrt{23}} \right) \sin \frac{5\sqrt{23}t}{4} \right] \\
&= -0.03e^{-15t/4} \left(\cos \frac{5\sqrt{23}t}{4} + \frac{3}{\sqrt{23}} \sin \frac{5\sqrt{23}t}{4} \right) \text{ m.}
\end{aligned}$$

42. The initial-value problem is

$$\frac{1}{5} \frac{d^2x}{dt^2} + \frac{3}{2} \frac{dx}{dt} + 10x = 4 \sin 10t \quad \implies \quad 2x'' + 15x' + 100x = 40 \sin 10t, \quad x(0) = 0, \quad x'(0) = 0.$$

Assuming that the solution satisfies the conditions of Corollary 'Corollary', we take Laplace transforms of both sides of the differential equation and use the initial conditions,

$$2[s^2X] + 15[sX] + 100X = \frac{400}{s^2 + 100}.$$

We solve this for the transform $X(s)$,

$$X(s) = \frac{400}{(s^2 + 100)(2s^2 + 15s + 100)}.$$

The inverse transform of this function is the solution of the initial-value problem

$$\begin{aligned}
x(t) &= \mathcal{L}^{-1} \left\{ \frac{400}{(s^2 + 100)(2s^2 + 15s + 100)} \right\} = \mathcal{L}^{-1} \left\{ \frac{-12s/65 - 80/65}{s^2 + 100} + \frac{24s/65 + 340/65}{2s^2 + 15s + 100} \right\} \\
&= -\frac{1}{65} (12 \cos 10t + 8 \sin 10t) + \frac{1}{65} \mathcal{L}^{-1} \left\{ \frac{12s + 170}{s^2 + 15s/2 + 50} \right\} \\
&= -\frac{1}{65} (12 \cos 10t + 8 \sin 10t) + \frac{1}{65} \mathcal{L}^{-1} \left\{ \frac{12(s + 15/4) + 125}{(s + 15/4)^2 + 575/16} \right\} \\
&= -\frac{1}{65} (12 \cos 10t + 8 \sin 10t) + \frac{e^{-15t/4}}{65} \mathcal{L}^{-1} \left\{ \frac{12s + 125}{s^2 + 575/16} \right\} \\
&= -\frac{1}{65} (12 \cos 10t + 8 \sin 10t) + \frac{e^{-15t/4}}{65} \left(12 \cos \frac{5\sqrt{23}t}{4} + \frac{100}{\sqrt{23}} \sin \frac{5\sqrt{23}t}{4} \right) \text{ m.}
\end{aligned}$$

43. The initial-value problem is

$$\frac{1}{10} \frac{d^2x}{dt^2} + \frac{1}{20} \frac{dx}{dt} + 5x = 0 \quad \implies \quad 2x'' + x' + 100x = 0, \quad x(0) = -\frac{1}{20}, \quad x'(0) = 2.$$

Assuming that the solution satisfies the conditions of Corollary 'Corollary', we take Laplace transforms of both sides of the differential equation and use the initial conditions,

$$2 \left[s^2X + \frac{s}{20} - 2 \right] + \left[sX + \frac{1}{20} \right] + 100X = 0.$$

We solve this for the transform $X(s)$,

$$X(s) = \frac{-s/10 + 79/20}{2s^2 + s + 100}.$$

The inverse transform of this function is the solution of the initial-value problem

$$\begin{aligned}
x(t) &= \mathcal{L}^{-1} \left\{ \frac{-s/10 + 79/20}{2s^2 + s + 100} \right\} = \frac{1}{40} \mathcal{L}^{-1} \left\{ \frac{-2s + 79}{s^2 + s/2 + 50} \right\} \\
&= \frac{1}{40} \mathcal{L}^{-1} \left\{ \frac{-2(s + 1/4) + 159/2}{(s + 1/4)^2 + 799/16} \right\} = \frac{e^{-t/4}}{40} \mathcal{L}^{-1} \left\{ \frac{-2s + 159/2}{s^2 + 799/16} \right\} \\
&= \frac{e^{-t/4}}{40} \left[-2 \cos \frac{\sqrt{799}t}{4} + \frac{159}{2} \left(\frac{4}{\sqrt{799}} \right) \sin \frac{\sqrt{799}t}{4} \right] \\
&= \frac{e^{-t/4}}{20} \left(\frac{159}{\sqrt{799}} \sin \frac{\sqrt{799}t}{4} - \cos \frac{\sqrt{799}t}{4} \right) \text{ m.}
\end{aligned}$$

44. The initial-value problem is

$$\frac{1}{10} \frac{d^2x}{dt^2} + 4000x = 3 \cos 200t \quad \implies \quad x'' + 40\,000x = 30 \cos 200t, \quad x(0) = 0, \quad x'(0) = 10.$$

Assuming that the solution satisfies the conditions of Corollary ‘Corollary’, we take Laplace transforms of both sides of the differential equation and use the initial conditions,

$$[s^2X - 10] + 40\,000X = \frac{30s}{s^2 + 40\,000}.$$

We solve this for the transform $X(s)$,

$$X(s) = \frac{10}{s^2 + 40\,000} + \frac{30s}{(s^2 + 40\,000)^2}.$$

The inverse transform of this function is the solution of the initial-value problem

$$\begin{aligned}
x(t) &= \mathcal{L}^{-1} \left\{ \frac{10}{s^2 + 40\,000} + \frac{30s}{(s^2 + 40\,000)^2} \right\} = \frac{10}{200} \sin 200t + \frac{30t}{400} \sin 200t \\
&= \frac{1}{20} \sin 200t + \frac{3t}{40} \sin 200t \text{ m.}
\end{aligned}$$

45. The initial-value problem is

$$\frac{d^2x}{dt^2} + 64x = 2 \sin 8t, \quad x(0) = 0, \quad x'(0) = 0.$$

Assuming that the solution satisfies the conditions of Corollary ‘Corollary’, we take Laplace transforms of both sides of the differential equation and use the initial conditions,

$$[s^2X] + 64X = \frac{16}{s^2 + 64}.$$

We solve this for the transform $X(s)$,

$$X(s) = \frac{16}{(s^2 + 64)^2}.$$

The inverse transform of this function is the solution of the initial-value problem

$$x(t) = \mathcal{L}^{-1} \left\{ \frac{16}{(s^2 + 64)^2} \right\} = \frac{16}{2(8)^3} (\sin 8t - 8t \cos 8t) = \frac{1}{64} (\sin 8t - 8t \cos 8t) \text{ m.}$$

46. (a) Since $\sin^2 at = (1 - \cos 2at)/2$,

$$\mathcal{L}\{\sin^2 at\} = \frac{1}{2} \left(\frac{1}{s} - \frac{s}{s^2 + 4a^2} \right) = \frac{2a^2}{s(s^2 + 4a^2)}.$$

(b) The derivative of $f(t) = \sin^2 at$ is $f'(t) = 2a \sin at \cos at = a \sin 2at$, and we know the Laplace transform of $\sin 2at$. We therefore take transforms of both sides of the equation and use equation 'Laplace transform for $f'(t)$ ' on the left,

$$sF(s) - 0 = \mathcal{L}\{a \sin 2at\} = \frac{2a^2}{s^2 + 4a^2} \implies F(s) = \frac{2a^2}{s(s^2 + 4a^2)}.$$

47. The first and second derivatives of $f(t) = t \cos at$ are

$$f'(t) = \cos at - at \sin at, \quad f''(t) = -a^2 t \cos at - 2a \sin at.$$

Since the second of these involves $\sin at$, whose transform we know, and the original function, we take Laplace transforms of this equation and use equation 'Laplace transform for $f''(t)$ ',

$$s^2 F(s) - 1 = -a^2 F(s) - \frac{2a^2}{s^2 + a^2} \implies (s^2 + a^2)F(s) = 1 - \frac{2a^2}{s^2 + a^2} = \frac{s^2 - a^2}{s^2 + a^2}.$$

$$\text{Thus, } \mathcal{L}\{t \cos at\} = \frac{s^2 - a^2}{(s^2 + a^2)^2}.$$

48. When $n = 0$, $t^0 = 1$, and

$$\mathcal{L}\{1\} = \int_0^\infty e^{-st} dt = \left\{ \frac{e^{-st}}{-s} \right\}_0^\infty = \frac{1}{s}, \quad \text{provided } s > 0.$$

The formula $\mathcal{L}\{t^n\} = n!/s^{n+1}$ is therefore valid for $n = 0$. Assume that it is valid for some integer k ; that is, assume that

$$\mathcal{L}\{t^k\} = \frac{k!}{s^{k+1}}.$$

If we take Laplace transforms of

$$\frac{d}{dt}(t^{k+1}) = (k+1)t^k,$$

and use equation 'Laplace transform for $f'(t)$ ', we get

$$s\mathcal{L}\{t^{k+1}\} = (k+1)\mathcal{L}\{t^k\} = (k+1) \left(\frac{k!}{s^{k+1}} \right) \implies \mathcal{L}\{t^{k+1}\} = \frac{(k+1)!}{s^{k+2}}.$$

Since this is the formula for $k+1$, we have proved that $\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$ for all $n \geq 0$.

49. (a) If we take Laplace transforms of $\frac{d}{dt}\sqrt{t} = \frac{1}{2\sqrt{t}}$, and use equation 'Laplace transform for $f'(t)$ ', we obtain

$$s\mathcal{L}\{\sqrt{t}\} = \mathcal{L}\left\{\frac{1}{2\sqrt{t}}\right\} = \frac{1}{2}\sqrt{\frac{\pi}{s}} \implies \mathcal{L}\{\sqrt{t}\} = \frac{1}{2s}\sqrt{\frac{\pi}{s}} = \frac{\sqrt{\pi}}{2s^{3/2}}.$$

(b) If we take Laplace transforms of $\frac{d}{dt}t^{3/2} = \frac{3}{2}\sqrt{t}$, and use equation 'Laplace transform for $f'(t)$ ', we obtain

$$s\mathcal{L}\{t^{3/2}\} = \mathcal{L}\left\{\frac{3}{2}\sqrt{t}\right\} = \frac{3}{2}\frac{\sqrt{\pi}}{2s^{3/2}}.$$

Thus, $\mathcal{L}\{t^{3/2}\} = \frac{3\sqrt{\pi}}{4s^{5/2}}$. If we now repeat this with $t^{5/2}$,

$$s\mathcal{L}\{t^{5/2}\} = \mathcal{L}\left\{\frac{5}{2}t^{3/2}\right\} = \frac{5}{2}\frac{3\sqrt{\pi}}{4s^{5/2}}.$$

Hence, $\mathcal{L}\{t^{5/2}\} = \frac{3 \cdot 5 \sqrt{\pi}}{8s^{7/2}}$. The pattern emerging is

$$\mathcal{L}\{t^{(2n+1)/2}\} = \frac{1 \cdot 3 \cdot 5 \cdots (2n+1)\sqrt{\pi}}{2^{n+1}s^{(2n+3)/2}} = \frac{(2n+1)!\sqrt{\pi}}{2^{2n+1}n!s^{(2n+3)/2}}.$$

This could be proved by mathematical induction.

50. (a) The proof is fashioned after that in Theorem ‘Theorem for Laplace transform of $f'(t)$ ’. Suppose that $t_j, j = 1, \dots, n$ denote the discontinuities of f' in $0 \leq t \leq T$. To ease notation, we shall assume that $t_0 < t_1$. Then,

$$\int_0^T e^{-st} f'(t) dt = \int_0^{t_0} e^{-st} f'(t) dt + \int_{t_0}^{t_1} e^{-st} f'(t) dt + \sum_{j=1}^n \int_{t_j}^{t_{j+1}} e^{-st} f'(t) dt.$$

Since f' is continuous on each subinterval, we may integrate by parts on these subintervals,

$$\begin{aligned} \int_0^T e^{-st} f'(t) dt &= \left[\{e^{-st} f(t)\}_0^{t_0} + s \int_0^{t_0} e^{-st} f(t) dt \right] + \left[\{e^{-st} f(t)\}_{t_0}^{t_1} + s \int_{t_0}^{t_1} e^{-st} f(t) dt \right] \\ &\quad + \sum_{j=1}^n \left[\{e^{-st} f(t)\}_{t_j}^{t_{j+1}} + s \int_{t_j}^{t_{j+1}} e^{-st} f(t) dt \right]. \end{aligned}$$

Because f is continuous at t_j for $j > 0$, it follows that $f(t_{j+}) = f(t_{j-}), j = 1, \dots, n$, and therefore

$$\begin{aligned} \int_0^T e^{-st} f'(t) dt &= \left[e^{-st_0} f(t_{0-}) - f(0) + s \int_0^{t_0} e^{-st} f(t) dt \right] + \left[-e^{-st_0} f(t_{0+}) + s \int_{t_0}^{t_1} e^{-st} f(t) dt \right] \\ &\quad + \sum_{j=1}^n s \int_{t_j}^{t_{j+1}} e^{-st} f(t) dt \\ &= -f(0) - e^{-st_0} [f(t_{0+}) - f(t_{0-})] + e^{-sT} f(T) + s \int_0^T e^{-st} f(t) dt. \end{aligned}$$

Thus,

$$\begin{aligned} \mathcal{L}\{f'\} &= \int_0^\infty e^{-st} f'(t) dt = \lim_{T \rightarrow \infty} \int_0^T e^{-st} f'(t) dt \\ &= \lim_{T \rightarrow \infty} \left[-f(0) - e^{-st_0} [f(t_{0+}) - f(t_{0-})] + e^{-sT} f(T) + s \int_0^T e^{-st} f(t) dt \right] \\ &= sF(s) - f(0) - e^{-st_0} [f(t_{0+}) - f(t_{0-})] + \lim_{T \rightarrow \infty} e^{-sT} f(T), \end{aligned}$$

provided the limit on the right exists. Since f is $O(e^{\alpha t})$, there exists M and \bar{T} such that for $t > \bar{T}$, $|f(t)| < Me^{\alpha t}$. Thus, for $T > \bar{T}$,

$$e^{-sT} |f(T)| < e^{-sT} M e^{\alpha T} = M e^{(\alpha-s)T}$$

which approaches 0 as $T \rightarrow \infty$ (provided $s > \alpha$). Consequently,

$$\mathcal{L}\{f'\} = sF(s) - f(0) - e^{-st_0} [f(t_{0+}) - f(t_{0-})].$$

(b) When $t_0 = 0$, the result becomes

$$\mathcal{L}\{f'\} = sF(s) - f(t_{0+}).$$

51. (a) Instead of using the definition for the convolution of t^n and t^m , we note that

$$\mathcal{L}\{t^n * t^m\} = \mathcal{L}\{t^n\} \mathcal{L}\{t^m\} = \left(\frac{n!}{s^{n+1}} \right) \left(\frac{m!}{s^{m+1}} \right) = \frac{n! m!}{s^{n+m+2}}.$$

When we take inverse transforms, we get

$$t^n * t^m = \frac{n! m!}{(n+m+1)!} t^{m+n+1}.$$

(b) When m and n are not positive integers, we use the result of Exercise 'Gamma function' in Section 'The Laplace Transform and its Inverse',

$$\mathcal{L}\{t^n * t^m\} = \mathcal{L}\{t^n\} \mathcal{L}\{t^m\} = \left[\frac{\Gamma(n+1)}{s^{n+1}} \right] \left[\frac{\Gamma(m+1)}{s^{m+1}} \right] = \frac{\Gamma(n+1)\Gamma(m+1)}{s^{n+m+2}}.$$

When we take inverse transforms, we get

$$t^n * t^m = \frac{\Gamma(n+1)\Gamma(m+1)}{\Gamma(n+m+2)} t^{m+n+1}.$$

52. When $f(t)$ satisfies the conditions of Theorem 'Theorem for Laplace transform of $\mathbf{f}'(\mathbf{t})$ ', the Laplace transform of $f'(t)$ exists for $s > \alpha$, and it is given by $\mathcal{L}\{f'(t)\} = sF(s) - f(0)$. According to Theorem 'initial-value theorem' of Section 'The Laplace Transform and its Inverse', the limit of $\mathcal{L}\{f'(t)\}$ must be zero as $s \rightarrow \infty$. This yields the required result.

53. If we take limits as $s \rightarrow 0$ on terms in equation 'Laplace transform for $\mathbf{f}'(\mathbf{t})$ ',

$$\lim_{s \rightarrow 0} \mathcal{L}\{f'(t)\} = \lim_{s \rightarrow 0} [sF(s)] - f(0).$$

Thus,

$$\lim_{s \rightarrow 0} [sF(s)] = \lim_{s \rightarrow 0} \left[\int_0^{\infty} e^{-st} f'(t) dt \right] + f(0) = \lim_{s \rightarrow 0} \left[\lim_{T \rightarrow \infty} \int_0^T e^{-st} f'(t) dt \right] + f(0).$$

If we interchange the order of limits, we get

$$\begin{aligned} \lim_{s \rightarrow 0} [sF(s)] &= \lim_{T \rightarrow \infty} \left[\lim_{s \rightarrow 0} \int_0^T e^{-st} f'(t) dt \right] + f(0) = \lim_{T \rightarrow \infty} \int_0^T f'(t) dt + f(0) \\ &= \lim_{T \rightarrow \infty} [f(T)] - f(0) + f(0) = \lim_{t \rightarrow \infty} f(t). \end{aligned}$$

54. The function $f(t) = \sin(e^{t^2})$ is continuous and of exponential order zero. Since its derivative $f'(t) = 2te^{t^2} \cos(e^{t^2})$ is continuous, Theorem 'Laplace transform for $\mathbf{f}'(\mathbf{t})$ ' guarantees that the Laplace transform of $f'(t)$ exists.

EXERCISES 6.4

1. (a) On the interval $0 < t < 1$, the differential equation $dy/dt + 3y = t$ has integrating factor e^{3t} , so that

$$\frac{d}{dt}(ye^{3t}) = te^{3t} \implies ye^{3t} = \frac{t}{3}e^{3t} - \frac{1}{9}e^{3t} + C \implies y(t) = \frac{t}{3} - \frac{1}{9} + Ce^{-3t}.$$

The initial condition requires $1 = y(0) = -1/9 + C$. Hence, for $0 < t < 1$, $y(t) = \frac{t}{3} - \frac{1}{9} + \frac{10}{9}e^{-3t}$. For $t > 1$, a general solution of the differential equation is $y(t) = 1/3 + De^{-3t}$. For the solution to be continuous at $t = 1$, we must have

$$\lim_{t \rightarrow 1^-} y(t) = \lim_{t \rightarrow 1^+} y(t) \implies \frac{1}{3} - \frac{1}{9} + \frac{10}{9}e^{-3} = \frac{1}{3} + De^{-3} \implies D = \frac{1}{9}(10 - e^3).$$

Thus,

$$y(t) = \begin{cases} \frac{t}{3} - \frac{1}{9} + \frac{10}{9}e^{-3t}, & 0 < t \leq 1 \\ \frac{1}{3} + \frac{1}{9}(10 - e^3)e^{-3t}, & t > 1. \end{cases}$$

- (b) When we write $f(t) = t[h(t) - h(t-1)] + h(t-1) = t + (1-t)h(t-1)$, and take Laplace transforms of the differential equation,

$$sY - 1 + 3Y = \frac{1}{s^2} + e^{-s}\mathcal{L}\{1 - (t+1)\} = \frac{1}{s^2} - \frac{e^{-s}}{s^2}.$$

Thus,

$$Y(s) = \frac{1}{s+3} \left(1 + \frac{1}{s^2} - \frac{e^{-s}}{s^2} \right).$$

Consequently,

$$\begin{aligned} y(t) &= \mathcal{L}^{-1} \left\{ \frac{1}{s+3} + \frac{1}{s^2(s+3)} - \frac{e^{-s}}{s^2(s+3)} \right\} \\ &= e^{-3t} + \mathcal{L}^{-1} \left\{ \left(\frac{1/3}{s^2} - \frac{1/9}{s} + \frac{1/9}{s+3} \right) - e^{-s} \left(\frac{1/3}{s^2} - \frac{1/9}{s} + \frac{1/9}{s+3} \right) \right\} \\ &= e^{-3t} + \frac{t}{3} - \frac{1}{9} + \frac{1}{9}e^{-3t} - \left[\frac{1}{3}(t-1) - \frac{1}{9} + \frac{1}{9}e^{-3(t-1)} \right] h(t-1) \\ &= \frac{t}{3} - \frac{1}{9} + \frac{10}{9}e^{-3t} + \left[-\frac{t}{3} + \frac{4}{9} - \frac{1}{9}e^{-3(t-1)} \right] h(t-1). \end{aligned}$$

2. When we take Laplace transforms of both sides of the differential equation and use the initial conditions,

$$[s^2Y - s(1) - 2] + 9Y = \mathcal{L}\{h(t-4)\} = \frac{e^{-4s}}{s}.$$

We solve this for the transform $Y(s)$,

$$Y(s) = \frac{s+2}{s^2+9} + \frac{e^{-4s}}{s(s^2+9)}.$$

The inverse transform of this function is the solution of the initial-value problem

$$\begin{aligned} y(t) &= \mathcal{L}^{-1} \left\{ \frac{s+2}{s^2+9} + \frac{e^{-4s}}{s(s^2+9)} \right\} = \mathcal{L}^{-1} \left\{ \frac{s+2}{s^2+9} + \left(\frac{1/9}{s} + \frac{-s/9}{s^2+9} \right) e^{-4s} \right\} \\ &= \cos 3t + \frac{2}{3} \sin 3t + \frac{1}{9} [1 - \cos 3(t-4)] h(t-4). \end{aligned}$$

3. When we take Laplace transforms of both sides of the differential equation and use the initial conditions,

$$[s^2Y - s(1) - 2] + 9Y = \mathcal{L}\{2[h(t) - h(t-4)]\} = 2\left(\frac{1}{s} - \frac{e^{-4s}}{s}\right).$$

We solve this for the transform $Y(s)$,

$$Y(s) = \frac{s+2}{s^2+9} + \frac{2(1-e^{-4s})}{s(s^2+9)}.$$

The inverse transform of this function is the solution of the initial-value problem

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}\left\{\frac{s+2}{s^2+9} + \frac{2(1-e^{-4s})}{s(s^2+9)}\right\} = \mathcal{L}^{-1}\left\{\frac{s+2}{s^2+9} + 2\left(\frac{1/9}{s} + \frac{-s/9}{s^2+9}\right)(1-e^{-4s})\right\} \\ &= \cos 3t + \frac{2}{3}\sin 3t + \frac{2}{9}(1 - \cos 3t) - \frac{2}{9}[1 - \cos 3(t-4)]h(t-4). \\ &= \frac{2}{9} + \frac{7}{9}\cos 3t + \frac{2}{3}\sin 3t - \frac{2}{9}[1 - \cos 3(t-4)]h(t-4). \end{aligned}$$

4. When we take Laplace transforms of both sides of the differential equation and use the initial conditions,

$$\begin{aligned} [s^2Y + 1] + 4[sY] + 4Y &= \mathcal{L}\{t[h(t) - h(t-1)] + h(t-1)\} = \mathcal{L}\{t + (1-t)h(t-1)\} = \frac{1}{s^2} + e^{-s}\mathcal{L}\{-t\} \\ &= \frac{1}{s^2} - \frac{e^{-s}}{s^2}. \end{aligned}$$

We solve this for the transform $Y(s)$,

$$Y(s) = \frac{-1}{s^2+4s+4} + \frac{1-e^{-s}}{s^2(s^2+4s+4)} = \frac{-1}{(s+2)^2} + \frac{1-e^{-s}}{s^2(s+2)^2}.$$

The inverse transform of this function is the solution of the initial-value problem

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}\left\{\frac{-1}{(s+2)^2} + \frac{1-e^{-s}}{s^2(s+2)^2}\right\} \\ &= \mathcal{L}^{-1}\left\{\frac{-1}{(s+2)^2} + \left[\frac{-1/4}{s} + \frac{1/4}{s^2} + \frac{1/4}{s+2} + \frac{1/4}{(s+2)^2}\right](1-e^{-s})\right\} \\ &= -te^{-2t} + \frac{1}{4}(-1+t+e^{-2t}+te^{-2t}) - \frac{1}{4}\left[-1+(t-1)+e^{-2(t-1)}+(t-1)e^{-2(t-1)}\right]h(t-1) \\ &= \frac{1}{4}(-1+t+e^{-2t}-3te^{-2t}) + \frac{1}{4}[2-t-te^{2(1-t)}]h(t-1). \end{aligned}$$

5. When we take Laplace transforms of both sides of the differential equation and use the initial conditions,

$$\begin{aligned} [s^2Y - s(-1)] + 4[sY + 1] + 4Y &= \mathcal{L}\{(2-t)[h(t) - h(t-2)] + (t-2)h(t-2)\} \\ &= \mathcal{L}\{2-t + 2(t-2)h(t-2)\} = \frac{2}{s} - \frac{1}{s^2} + 2e^{-2s}\mathcal{L}\{t\} \\ &= \frac{2}{s} - \frac{1}{s^2} + \frac{2e^{-2s}}{s^2}. \end{aligned}$$

We solve this for the transform $Y(s)$,

$$Y(s) = -\frac{s+4}{s^2+4s+4} + \frac{2}{s(s^2+4s+4)} - \frac{1-2e^{-2s}}{s^2(s^2+4s+4)} = -\frac{s+4}{(s+2)^2} + \frac{2}{s(s+2)^2} - \frac{1-2e^{-2s}}{s^2(s+2)^2}.$$

The inverse transform of this function is the solution of the initial-value problem

$$\begin{aligned}
y(t) &= \mathcal{L}^{-1} \left\{ -\frac{s+4}{(s+2)^2} + \frac{2}{s(s+2)^2} - \frac{1-2e^{-2s}}{s^2(s+2)^2} \right\} \\
&= \mathcal{L}^{-1} \left\{ \frac{1/2}{s} - \frac{3/2}{s+2} - \frac{3}{(s+2)^2} - \left[\frac{-1/4}{s} + \frac{1/4}{s^2} + \frac{1/4}{s+2} + \frac{1/4}{(s+2)^2} \right] (1-2e^{-2s}) \right\} \\
&= \frac{1}{2} - \frac{3}{2}e^{-2t} - 3te^{-2t} + \frac{1}{4}(1-t-e^{-2t}-te^{-2t}) \\
&\quad + \frac{1}{2}[-1+(t-2)+e^{-2(t-2)}+(t-2)e^{-2(t-2)}]h(t-2) \\
&= \frac{3}{4} - \frac{t}{4} - \frac{7}{4}e^{-2t} - \frac{13t}{4}e^{-2t} + \frac{1}{2}[-3+t-e^{2(2-t)}+te^{2(2-t)}]h(t-2).
\end{aligned}$$

6. When we take Laplace transforms of both sides of the differential equation and use the initial conditions,

$$\begin{aligned}
[s^2Y - s(1) - 2] + 4[sY - 1] + 3Y &= \mathcal{L}\{\sin t h(t - \pi)\} = e^{-\pi s} \mathcal{L}\{\sin(t + \pi)\} = e^{-\pi s} \mathcal{L}\{-\sin t\} \\
&= -\frac{e^{-\pi s}}{s^2 + 1}.
\end{aligned}$$

We solve this for the transform $Y(s)$,

$$Y(s) = \frac{s+6}{s^2+4s+3} - \frac{e^{-\pi s}}{(s^2+1)(s^2+4s+3)} = \frac{s+6}{(s+1)(s+3)} - \frac{e^{-\pi s}}{(s^2+1)(s+1)(s+3)}.$$

The inverse transform of this function is the solution of the initial-value problem

$$\begin{aligned}
y(t) &= \mathcal{L}^{-1} \left\{ \frac{s+6}{(s+1)(s+3)} - \frac{e^{-\pi s}}{(s^2+1)(s+1)(s+3)} \right\} \\
&= \mathcal{L}^{-1} \left\{ \frac{5/2}{s+1} - \frac{3/2}{s+3} - \left(\frac{1/4}{s+1} - \frac{1/20}{s+3} + \frac{-s/5+1/10}{s^2+1} \right) e^{-\pi s} \right\} \\
&= \frac{5}{2}e^{-t} - \frac{3}{2}e^{-3t} + \frac{1}{20} \left[-5e^{-(t-\pi)} + e^{-3(t-\pi)} + 4\cos(t-\pi) - 2\sin(t-\pi) \right] h(t-\pi) \\
&= \frac{5}{2}e^{-t} - \frac{3}{2}e^{-3t} + \frac{1}{20} \left[-5e^{\pi-t} + e^{3(\pi-t)} - 4\cos t + 2\sin t \right] h(t-\pi).
\end{aligned}$$

7. When we take Laplace transforms of both sides of the differential equation and use the initial conditions,

$$\begin{aligned}
[s^2Y - s(1) - 2] + 4[sY - 1] + 3Y &= \mathcal{L}\{\sin t[h(t) - h(t-\pi)]\} = \frac{1}{s^2+1} - e^{-\pi s} \mathcal{L}\{\sin(t+\pi)\} \\
&= \frac{1}{s^2+1} - e^{-\pi s} \mathcal{L}\{-\sin t\} = \frac{1}{s^2+1} + \frac{e^{-\pi s}}{s^2+1}.
\end{aligned}$$

We solve this for the transform $Y(s)$,

$$Y(s) = \frac{s+6}{s^2+4s+3} + \frac{1+e^{-\pi s}}{(s^2+1)(s^2+4s+3)} = \frac{s+6}{(s+1)(s+3)} + \frac{1+e^{-\pi s}}{(s^2+1)(s+1)(s+3)}.$$

The inverse transform of this function is the solution of the initial-value problem

$$\begin{aligned}
y(t) &= \mathcal{L}^{-1} \left\{ \frac{s+6}{(s+1)(s+3)} + \frac{1+e^{-\pi s}}{(s^2+1)(s+1)(s+3)} \right\} \\
&= \mathcal{L}^{-1} \left\{ \frac{5/2}{s+1} - \frac{3/2}{s+3} + \left(\frac{1/4}{s+1} - \frac{1/20}{s+3} + \frac{-s/5+1/10}{s^2+1} \right) (1+e^{-\pi s}) \right\} \\
&= \frac{5}{2}e^{-t} - \frac{3}{2}e^{-3t} + \frac{1}{20} [5e^{-t} - e^{-3t} - 4\cos t + 2\sin t] \\
&\quad + \frac{1}{20} [5e^{-(t-\pi)} - e^{-3(t-\pi)} - 4\cos(t-\pi) + 2\sin(t-\pi)] h(t-\pi) \\
&= \frac{11}{4}e^{-t} - \frac{31}{20}e^{-3t} - \frac{1}{5}\cos t + \frac{1}{10}\sin t + \frac{1}{20} [5e^{\pi-t} - e^{3(\pi-t)} + 4\cos t - 2\sin t] h(t-\pi).
\end{aligned}$$

8. When we take Laplace transforms of both sides of the differential equation and use the initial conditions,

$$[s^2Y] + 2[sY] + 5Y = \mathcal{L}\{3[h(t) - h(t-1)] - 3h(t-1)\} = \frac{3}{s} - 6\mathcal{L}\{h(t-1)\} = \frac{3}{s} - \frac{6e^{-s}}{s}.$$

We solve this for the transform $Y(s)$,

$$Y(s) = \frac{3(1 - 2e^{-s})}{s(s^2 + 2s + 5)}.$$

The inverse transform of this function is the solution of the initial-value problem

$$\begin{aligned} y(t) &= \mathcal{L}^{-1} \left\{ \frac{3(1 - 2e^{-s})}{s(s^2 + 2s + 5)} \right\} = 3\mathcal{L}^{-1} \left\{ \left(\frac{1/5}{s} - \frac{s/5 + 2/5}{s^2 + 2s + 5} \right) (1 - 2e^{-s}) \right\} \\ &= \frac{3}{5} \mathcal{L}^{-1} \left\{ \left[\frac{1}{s} - \frac{(s+1)+1}{(s+1)^2 + 4} \right] (1 - 2e^{-s}) \right\} \\ &= \frac{3}{5} \left[1 - e^{-t} \left(\cos 2t + \frac{1}{2} \sin 2t \right) \right] - \frac{6}{5} \left\{ 1 - e^{-(t-1)} \left[\cos 2(t-1) + \frac{1}{2} \sin 2(t-1) \right] \right\} h(t-1). \end{aligned}$$

9. When we take Laplace transforms of both sides of the differential equation and use the initial conditions,

$$\begin{aligned} [s^2Y] + 2[sY] + 5Y &= \mathcal{L}\{4[h(t) - h(t-1)] - 4[h(t-1) - h(t-2)]\} = \mathcal{L}\{4 - 8h(t-1) + 4h(t-2)\} \\ &= \frac{4}{s} - \frac{8e^{-s}}{s} + \frac{4e^{-2s}}{s}. \end{aligned}$$

We solve this for the transform $Y(s)$,

$$Y(s) = \frac{4}{s(s^2 + 2s + 5)}(1 - 2e^{-s} + e^{-2s}).$$

The inverse transform of this function is the solution of the initial-value problem

$$\begin{aligned} y(t) &= \mathcal{L}^{-1} \left\{ \frac{4}{s(s^2 + 2s + 5)}(1 - 2e^{-s} + e^{-2s}) \right\} = 4\mathcal{L}^{-1} \left\{ \left(\frac{1/5}{s} - \frac{s/5 + 2/5}{s^2 + 2s + 5} \right) (1 - 2e^{-s} + e^{-2s}) \right\} \\ &= \frac{4}{5} \mathcal{L}^{-1} \left\{ \left[\frac{1}{s} - \frac{(s+1)+1}{(s+1)^2 + 4} \right] (1 - 2e^{-s} + e^{-2s}) \right\} \\ &= \frac{4}{5} \left[1 - e^{-t} \left(\cos 2t + \frac{1}{2} \sin 2t \right) \right] - \frac{8}{5} \left\{ 1 - e^{-(t-1)} \left[\cos 2(t-1) + \frac{1}{2} \sin 2(t-1) \right] \right\} h(t-1) \\ &\quad + \frac{4}{5} \left\{ 1 - e^{-(t-2)} \left[\cos 2(t-2) + \frac{1}{2} \sin 2(t-2) \right] \right\} h(t-2) \\ &= \frac{2}{5} [2 - e^{-t} (2 \cos 2t + \sin 2t)] + \frac{4}{5} \{ -2 + e^{1-t} [2 \cos 2(t-1) + \sin 2(t-1)] \} h(t-1) \\ &\quad + \frac{2}{5} \{ 2 - e^{2-t} [2 \cos 2(t-2) + \sin 2(t-2)] \} h(t-2). \end{aligned}$$

10. When we take Laplace transforms of both sides of the differential equation and use the initial conditions,

$$\begin{aligned} [s^2Y - s(2)] + 16Y &= \frac{1}{1 - e^{-2s}} \mathcal{L}\{t[h(t) - h(t-1)]\} = \frac{1}{1 - e^{-2s}} \left[\frac{1}{s^2} - e^{-s} \mathcal{L}\{t+1\} \right] \\ &= \frac{1}{1 - e^{-2s}} \left[\frac{1}{s^2} - e^{-s} \left(\frac{1}{s^2} + \frac{1}{s} \right) \right] = \frac{1}{(1 + e^{-s})(1 - e^{-s})} \left[\frac{1}{s^2} (1 - e^{-s}) - \frac{e^{-s}}{s} \right]. \end{aligned}$$

We solve this for the transform $Y(s)$,

$$Y(s) = \frac{2s}{s^2 + 16} + \frac{1}{s^2(s^2 + 16)(1 + e^{-s})} - \frac{e^{-s}}{s(s^2 + 16)(1 - e^{-2s})}.$$

The inverse transform of this function is the solution of the initial-value problem

$$\begin{aligned} y(t) &= \mathcal{L}^{-1} \left\{ \frac{2s}{s^2 + 16} + \frac{1}{s^2(s^2 + 16)(1 + e^{-s})} - \frac{e^{-s}}{s(s^2 + 16)(1 - e^{-2s})} \right\} \\ &= \mathcal{L}^{-1} \left\{ \frac{2s}{s^2 + 16} + \left(\frac{1/16}{s^2} - \frac{1/16}{s^2 + 16} \right) \sum_{n=0}^{\infty} (-1)^n e^{-ns} - \left(\frac{1/16}{s} - \frac{s/16}{s^2 + 16} \right) e^{-s} \sum_{n=0}^{\infty} e^{-2ns} \right\} \\ &= \mathcal{L}^{-1} \left\{ \frac{2s}{s^2 + 16} + \frac{1}{16} \left(\frac{1}{s^2} - \frac{1}{s^2 + 16} \right) \sum_{n=0}^{\infty} (-1)^n e^{-ns} - \frac{1}{16} \left(\frac{1}{s} - \frac{s}{s^2 + 16} \right) \sum_{n=0}^{\infty} e^{-(2n+1)s} \right\} \\ &= 2 \cos 4t + \frac{1}{16} \sum_{n=0}^{\infty} (-1)^n \left[(t - n) - \frac{1}{4} \sin 4(t - n) \right] h(t - n) \\ &\quad - \frac{1}{16} \sum_{n=0}^{\infty} [1 - \cos 4(t - 2n - 1)] h(t - 2n - 1) \\ &= 2 \cos 4t + \frac{1}{64} \sum_{n=0}^{\infty} (-1)^n [4(t - n) - \sin 4(t - n)] h(t - n) \\ &\quad - \frac{1}{16} \sum_{n=0}^{\infty} [1 - \cos 4(t - 2n - 1)] h(t - 2n - 1). \end{aligned}$$

11. When we take Laplace transforms of both sides of the differential equation and use the initial conditions,

$$\begin{aligned} [s^2 Y - s(2)] + 16Y &= \frac{1}{1 - e^{-2s}} \mathcal{L}\{t[h(t) - h(t - 1)] + (2 - t)[h(t - 1) - h(t - 2)]\} \\ &= \frac{1}{1 - e^{-2s}} \mathcal{L}\{t + (2 - 2t)h(t - 1) + (t - 2)h(t - 2)\} \\ &= \frac{1}{1 - e^{-2s}} \left[\frac{1}{s^2} + e^{-s} \mathcal{L}\{2 - 2(t + 1)\} + e^{-2s} \mathcal{L}\{t\} \right] \\ &= \frac{1}{1 - e^{-2s}} \left[\frac{1}{s^2} - \frac{2e^{-s}}{s^2} + \frac{e^{-2s}}{s^2} \right] = \frac{(1 - e^{-s})^2}{s^2(1 - e^{-s})(1 + e^{-s})} = \frac{1 - e^{-s}}{s^2(1 + e^{-s})}. \end{aligned}$$

We solve this for the transform $Y(s)$,

$$Y(s) = \frac{2s}{s^2 + 16} + \frac{1 - e^{-s}}{s^2(s^2 + 16)(1 + e^{-s})}.$$

The inverse transform of this function is the solution of the initial-value problem

$$\begin{aligned} y(t) &= \mathcal{L}^{-1} \left\{ \frac{2s}{s^2 + 16} + \frac{1 - e^{-s}}{s^2(s^2 + 16)(1 + e^{-s})} \right\} \\ &= \mathcal{L}^{-1} \left\{ \frac{2s}{s^2 + 16} + \left(\frac{1/16}{s^2} - \frac{1/16}{s^2 + 16} \right) (1 - e^{-s}) \sum_{n=0}^{\infty} (-1)^n e^{-ns} \right\} \\ &= \mathcal{L}^{-1} \left\{ \frac{2s}{s^2 + 16} + \frac{1}{16} \left(\frac{1}{s^2} - \frac{1}{s^2 + 16} \right) \left[\sum_{n=0}^{\infty} (-1)^n e^{-ns} + \sum_{n=0}^{\infty} (-1)^{n+1} e^{-(n+1)s} \right] \right\} \\ &= 2 \cos 4t + \frac{1}{16} \sum_{n=0}^{\infty} (-1)^n \left[(t - n) - \frac{1}{4} \sin 4(t - n) \right] h(t - n) \\ &\quad + \frac{1}{16} \sum_{n=0}^{\infty} (-1)^{n+1} \left[(t - n - 1) - \frac{1}{4} \sin 4(t - n - 1) \right] h(t - n - 1) \end{aligned}$$

$$\begin{aligned}
&= 2 \cos 4t + \frac{1}{64} \sum_{n=0}^{\infty} (-1)^n [4(t-n) - \sin 4(t-n)] h(t-n) \\
&\quad + \frac{1}{64} \sum_{n=0}^{\infty} (-1)^{n+1} [4(t-n-1) - \sin 4(t-n-1)] h(t-n-1).
\end{aligned}$$

12. When we take Laplace transforms of both sides of the differential equation and use the initial conditions,

$$\begin{aligned}
s^2 Y + sY &= \frac{1}{1 - e^{-2s}} \mathcal{L}\{[h(t) - h(t-1)] - [h(t-1) - h(t-2)]\} \\
&= \frac{1}{1 - e^{-2s}} \left(\frac{1}{s} - \frac{2e^{-s}}{s} + \frac{e^{-2s}}{s} \right) = \frac{(1 - e^{-s})^2}{s(1 - e^{-s})(1 + e^{-s})} = \frac{1 - e^{-s}}{s(1 + e^{-s})}.
\end{aligned}$$

Thus,

$$\begin{aligned}
Y(s) &= \frac{1 - e^{-s}}{s^2(s+1)(1 + e^{-s})} = \left(\frac{1}{s^2} - \frac{1}{s} + \frac{1}{s+1} \right) (1 - e^{-s}) \sum_{n=0}^{\infty} (-e^{-s})^n \\
&= \sum_{n=0}^{\infty} (-1)^n \left(\frac{1}{s^2} - \frac{1}{s} + \frac{1}{s+1} \right) [e^{-ns} - e^{(n+1)s}].
\end{aligned}$$

Inverse transforms now give

$$y(t) = \sum_{n=0}^{\infty} (-1)^n \left\{ [(t-n-1) + e^{-(t-n)}] h(t-n) - [(t-n-2) + e^{-(t-n-1)}] h(t-n-1) \right\}.$$

13. The initial-value problem for the number of grams $g(t)$ of salt in the tank as a function of time t in seconds is

$$\frac{dg}{dt} + \frac{g}{2 \times 10^5} = \frac{1}{10}, \quad g(0) = 5000.$$

When we take Laplace transforms,

$$sG - 5000 + \frac{G}{2 \times 10^5} = \frac{1}{10s}.$$

We solve this for $G(s)$,

$$\begin{aligned}
G(s) &= \frac{5000}{s + \frac{1}{2 \times 10^5}} + \frac{1}{10s \left(s + \frac{1}{2 \times 10^5} \right)} \\
&= \frac{5000}{s + \frac{1}{2 \times 10^5}} + \frac{1}{10} \left(\frac{2 \times 10^5}{s} - \frac{2 \times 10^5}{s + \frac{1}{2 \times 10^5}} \right) = \frac{20000}{s} - \frac{15000}{s + \frac{1}{200000}}.
\end{aligned}$$

Inverse transforms now give

$$g(t) = 20000 - 15000e^{-t/200000}.$$

14. Laplace transforms cannot handle the quotient $S(t)/(10^6 + 5t)$.
 15. The initial-value problem for the number of grams $g(t)$ of salt in the tank as a function of time t in seconds is

$$\frac{dg}{dt} + \frac{g}{2 \times 10^5} = \frac{1}{10}[h(t) - h(t-600)] + \frac{1}{20}h(t-600), \quad g(0) = 5000.$$

When we take Laplace transforms,

$$sG - 5000 + \frac{G}{2 \times 10^5} = \frac{1}{10} \left(\frac{1}{s} - \frac{e^{-600s}}{s} \right) + \frac{e^{-600s}}{20s}.$$

We solve this for $G(s)$,

$$\begin{aligned} G(s) &= \frac{5000}{s + \frac{1}{2 \times 10^5}} + \frac{\frac{1}{20} \left(\frac{2}{s} - \frac{e^{-600s}}{s} \right)}{s + \frac{1}{2 \times 10^5}} \\ &= \frac{5000}{s + \frac{1}{2 \times 10^5}} + \frac{1}{20} (2 - e^{-600s}) \left(\frac{2 \times 10^5}{s} - \frac{2 \times 10^5}{s + \frac{1}{2 \times 10^5}} \right) \\ &= \frac{20\,000}{s} - \frac{15\,000}{s + \frac{1}{200\,000}} - 10\,000 e^{-600s} \left(\frac{1}{s} - \frac{1}{s + \frac{1}{200\,000}} \right). \end{aligned}$$

Inverse transforms now give

$$g(t) = 20\,000 - 15\,000 e^{-t/200\,000} - 10\,000 \left[1 - e^{-(t-600)/200\,000} \right] h(t-600).$$

16. The initial-value problem for the number of grams $g(t)$ of salt in the tank is

$$\frac{dg}{dt} = f(t) - \frac{5g}{10^6}, \quad g(0) = 5000,$$

where

$$f(t) = \begin{cases} 1/10, & 0 < t < 60 \\ 0, & 60 < t < 120, \end{cases} \quad f(t+120) = f(t).$$

When we take Laplace transforms,

$$\begin{aligned} sG - 5000 &= \frac{1}{1 - e^{-120s}} \mathcal{L} \left\{ \frac{1}{10} [h(t) - h(t-60)] \right\} - \frac{5G}{10^6} \\ &= \frac{1}{10(1 - e^{-120s})} \left(\frac{1}{s} - \frac{e^{-60s}}{s} \right) - \frac{5G}{10^6}. \end{aligned}$$

Solving for $G(s)$ gives

$$\begin{aligned} G(s) &= \frac{1 - e^{-60s}}{10s(s + 5/10^6)(1 - e^{-120s})} + \frac{5000}{s + 5/10^6} \\ &= \frac{1}{10s(s + 5/10^6)(1 + e^{-60s})} + \frac{5000}{s + 5/10^6} \\ &= \frac{10^5}{5} \left(\frac{1}{s} - \frac{1}{s + 5/10^6} \right) \sum_{n=0}^{\infty} (-1)^n e^{-60ns} + \frac{5000}{s + 5/10^6}. \end{aligned}$$

Thus,

$$g(t) = 20\,000 \sum_{n=0}^{\infty} (-1)^n [1 - e^{-5(t-60n)/10^6}] h(t-60n) + 5000 e^{-5t/10^6}.$$

17. The initial-value problem for the number of grams $g(t)$ of glucose in the bloodstream is

$$\frac{dg}{dt} = R[h(t) - h(t-240)] - kg, \quad g(0) = g_0,$$

where $k > 0$ is a constant. When we take Laplace transforms,

$$sG - g_0 = \frac{R}{s} (1 - e^{-240s}) - kG.$$

We now solve for $G(s)$,

$$G(s) = \frac{R}{s(s+k)} (1 - e^{-240s}) + \frac{g_0}{s+k} = \frac{R}{k} \left(\frac{1}{s} - \frac{1}{s+k} \right) (1 - e^{-240s}) + \frac{g_0}{s+k}.$$

Inverse transforms give

$$g(t) = \frac{R}{k}(1 - e^{-kt}) - \frac{R}{k}[1 - e^{-k(t-240)}]h(t-240) + g_0e^{-kt}.$$

After 6 hours, the number of grams of glucose in the bloodstream is

$$g(360) = \frac{R}{k}(1 - e^{-360k}) - \frac{R}{k}(1 - e^{-120k}) + g_0e^{-360k} = \frac{R}{k}(e^{-120k} - e^{-360k}) + g_0e^{-360k}.$$

18. The initial-value problem for the number of grams $g(t)$ of glucose in the bloodstream is

$$\frac{dg}{dt} = Rf(t) - kg, \quad g(0) = g_0,$$

where $k > 0$ is a constant, and $f(t)$ is the periodic function

$$f(t) = h(t) - h(t-60) = 1 - h(t-60), \quad 0 < t < 120, \quad f(t+120) = f(t).$$

When we take Laplace transforms,

$$sG - g_0 = \frac{R}{1 - e^{-120s}}\mathcal{L}\{1 - h(t-60)\} - kG = \frac{R}{s(1 - e^{-120s})}(1 - e^{-60s}) - kG.$$

We now solve for $G(s)$,

$$G(s) = \frac{R}{s(s+k)(1 + e^{-60s})} + \frac{g_0}{s+k} = \frac{R}{k} \left(\frac{1}{s} - \frac{1}{s+k} \right) \sum_{n=0}^{\infty} (-1)^n e^{-60ns} + \frac{g_0}{s+k}.$$

Inverse transforms give

$$g(t) = \frac{R}{k} \sum_{n=0}^{\infty} (-1)^n [1 - e^{-k(t-60n)}]h(t-60n) + g_0e^{-kt}.$$

19. (a) If time t is measured in months, the initial-value problem for $m(t)$ is

$$\frac{dm}{dt} = km - H[h(t) - h(t-1)], \quad m(0) = m_0.$$

If we replace $h(t)$ with 1, and take Laplace transforms,

$$sM - m_0 = kM - H \left(\frac{1}{s} - \frac{e^{-s}}{s} \right).$$

When we solve for $M(s)$,

$$M(s) = \frac{m_0}{s-k} - \frac{H(1 - e^{-s})}{s(s-k)} = \frac{m_0}{s-k} + \frac{H}{k} \left(\frac{1}{s} - \frac{1}{s-k} \right) (1 - e^{-s}).$$

Inverse transforms give

$$m(t) = m_0e^{kt} + \frac{H}{k} \left[(1 - e^{kt}) - (1 - e^{k(t-1)})h(t-1) \right] \text{ kg.}$$

- (b) For $m(12) = m_0$,

$$m_0 = m_0e^{12k} + \frac{H}{k} [(1 - e^{12k}) - (1 - e^{11k})] \implies H = \frac{km_0(e^{12k} - 1)}{e^{12k} - e^{11k}}.$$

20. The initial-value problem for displacement of the mass from its equilibrium position is

$$\frac{1}{10} \frac{d^2x}{dt^2} + 40x = 100[h(t) - h(t-4)], \quad x(0) = \frac{1}{10}, \quad x'(0) = -2.$$

We write the differential equation in the form

$$\frac{d^2x}{dt^2} + 400x = 1000[1 - h(t - 4)],$$

and take Laplace transforms,

$$\left[s^2X - \frac{s}{10} + 2\right] + 400X = 1000\left(\frac{1}{s} - \frac{e^{-4s}}{s}\right) = \frac{1000(1 - e^{-4s})}{s}.$$

We solve this for the transform $X(s)$,

$$X(s) = \frac{s/10 - 2}{s^2 + 400} + \frac{1000(1 - e^{-4s})}{s(s^2 + 400)}.$$

The inverse transform of this function is the solution of the initial-value problem

$$\begin{aligned} x(t) &= \mathcal{L}^{-1}\left\{\frac{s/10 - 2}{s^2 + 400} + \frac{1000(1 - e^{-4s})}{s(s^2 + 400)}\right\} \\ &= \mathcal{L}^{-1}\left\{\frac{s/10 - 2}{s^2 + 400} + \frac{5}{2}\left(\frac{1}{s} - \frac{s}{s^2 + 400}\right)(1 - e^{-4s})\right\} \\ &= \frac{1}{10}\cos 20t - \frac{1}{10}\sin 20t + \frac{5}{2}(1 - \cos 20t) - \frac{5}{2}[1 - \cos 20(t - 4)]h(t - 4) \\ &= \frac{5}{2} - \frac{12}{5}\cos 20t - \frac{1}{10}\sin 20t - \frac{5}{2}[1 - \cos 20(t - 4)]h(t - 4) \text{ m.} \end{aligned}$$

21. The initial-value problem for displacement of the mass from its equilibrium position is

$$\frac{1}{10}\frac{d^2x}{dt^2} + 40x = 100h(t - 4), \quad x(0) = \frac{1}{10}, \quad x'(0) = -2.$$

We write the differential equation in the form

$$\frac{d^2x}{dt^2} + 400x = 1000h(t - 4),$$

and take Laplace transforms,

$$\left[s^2X - \frac{s}{10} + 2\right] + 400X = \frac{1000e^{-4s}}{s}.$$

We solve this for the transform $X(s)$,

$$X(s) = \frac{s/10 - 2}{s^2 + 400} + \frac{1000e^{-4s}}{s(s^2 + 400)}.$$

The inverse transform of this function is the solution of the initial-value problem

$$\begin{aligned} x(t) &= \mathcal{L}^{-1}\left\{\frac{s/10 - 2}{s^2 + 400} + \frac{1000e^{-4s}}{s(s^2 + 400)}\right\} \\ &= \mathcal{L}^{-1}\left\{\frac{s/10 - 2}{s^2 + 400} + \frac{5}{2}\left(\frac{1}{s} - \frac{s}{s^2 + 400}\right)e^{-4s}\right\} \\ &= \frac{1}{10}\cos 20t - \frac{1}{10}\sin 20t + \frac{5}{2}[1 - \cos 20(t - 4)]h(t - 4) \text{ m.} \end{aligned}$$

22. The initial-value problem for displacement of the mass from its equilibrium position is

$$\frac{1}{10}\frac{d^2x}{dt^2} + 5\frac{dx}{dt} + 40x = 100[h(t) - h(t - 4)], \quad x(0) = \frac{1}{10}, \quad x'(0) = -2.$$

We write the differential equation in the form

$$\frac{d^2x}{dt^2} + 50\frac{dx}{dt} + 400x = 1000[1 - h(t - 4)],$$

and take Laplace transforms,

$$\left[s^2X - \frac{s}{10} + 2\right] + 50\left[sX - \frac{1}{10}\right] + 400X = 1000\left(\frac{1}{s} - \frac{e^{-4s}}{s}\right).$$

We solve this for the transform $X(s)$,

$$X(s) = \frac{s/10 + 3}{s^2 + 50s + 400} + \frac{1000(1 - e^{-4s})}{s(s^2 + 50s + 400)} = \frac{s/10 + 3}{(s + 10)(s + 40)} + \frac{1000(1 - e^{-4s})}{s(s + 10)(s + 40)}.$$

The inverse transform of this function is the solution of the initial-value problem

$$\begin{aligned} x(t) &= \mathcal{L}^{-1}\left\{\frac{s/10 + 3}{(s + 10)(s + 40)} + \frac{1000(1 - e^{-4s})}{s(s + 10)(s + 40)}\right\} \\ &= \mathcal{L}^{-1}\left\{\frac{1/15}{s + 10} + \frac{1/30}{s + 40} + 1000\left(\frac{1/400}{s} - \frac{1/300}{s + 10} + \frac{1/1200}{s + 40}\right)(1 - e^{-4s})\right\} \\ &= \frac{1}{15}e^{-10t} + \frac{1}{30}e^{-40t} + \frac{5}{2} - \frac{10}{3}e^{-10t} + \frac{5}{6}e^{-40t} - \left[\frac{5}{2} - \frac{10}{3}e^{-10(t-4)} + \frac{5}{6}e^{-40(t-4)}\right]h(t-4) \\ &= \frac{5}{2} - \frac{49}{15}e^{-10t} + \frac{13}{15}e^{-40t} - \frac{5}{6}\left[3 - 4e^{10(4-t)} + e^{40(4-t)}\right]h(t-4) \text{ m.} \end{aligned}$$

- 23.** The initial-value problem for displacement of the mass from its equilibrium position is

$$\frac{1}{10}\frac{d^2x}{dt^2} + 5\frac{dx}{dt} + 40x = 100h(t-4), \quad x(0) = \frac{1}{10}, \quad x'(0) = -2.$$

We write the differential equation in the form

$$\frac{d^2x}{dt^2} + 50\frac{dx}{dt} + 400x = 1000h(t-4),$$

and take Laplace transforms,

$$\left[s^2X - \frac{s}{10} + 2\right] + 50\left[sX - \frac{1}{10}\right] + 400X = \frac{1000e^{-4s}}{s}.$$

We solve this for the transform $X(s)$,

$$X(s) = \frac{s/10 + 3}{s^2 + 50s + 400} + \frac{1000e^{-4s}}{s(s^2 + 50s + 400)} = \frac{s/10 + 3}{(s + 10)(s + 40)} + \frac{1000e^{-4s}}{s(s + 10)(s + 40)}.$$

The inverse transform of this function is the solution of the initial-value problem

$$\begin{aligned} x(t) &= \mathcal{L}^{-1}\left\{\frac{s/10 + 3}{(s + 10)(s + 40)} + \frac{1000e^{-4s}}{s(s + 10)(s + 40)}\right\} \\ &= \mathcal{L}^{-1}\left\{\frac{1/15}{s + 10} + \frac{1/30}{s + 40} + 1000\left(\frac{1/400}{s} - \frac{1/300}{s + 10} + \frac{1/1200}{s + 40}\right)e^{-4s}\right\} \\ &= \frac{1}{15}e^{-10t} + \frac{1}{30}e^{-40t} + \left[\frac{5}{2} - \frac{10}{3}e^{-10(t-4)} + \frac{5}{6}e^{-40(t-4)}\right]h(t-4) \\ &= \frac{1}{15}e^{-10t} + \frac{1}{30}e^{-40t} + \frac{5}{6}\left[3 - 4e^{10(4-t)} + e^{40(4-t)}\right]h(t-4) \text{ m.} \end{aligned}$$

- 24.** The initial-value problem for displacement of the mass from its equilibrium position is

$$\frac{1}{10}\frac{d^2x}{dt^2} + \frac{dx}{dt} + 40x = 100[h(t) - h(t-4)], \quad x(0) = \frac{1}{10}, \quad x'(0) = -2.$$

We write the differential equation in the form

$$\frac{d^2x}{dt^2} + 10\frac{dx}{dt} + 400x = 1000[1 - h(t-4)],$$

and take Laplace transforms,

$$\left[s^2X - \frac{s}{10} + 2\right] + 10\left[sX - \frac{1}{10}\right] + 400X = 1000\left(\frac{1}{s} - \frac{e^{-4s}}{s}\right).$$

We solve this for the transform $X(s)$,

$$X(s) = \frac{s/10 - 1}{s^2 + 10s + 400} + \frac{1000(1 - e^{-4s})}{s(s^2 + 10s + 400)}.$$

The inverse transform of this function is the solution of the initial-value problem

$$\begin{aligned} x(t) &= \mathcal{L}^{-1}\left\{\frac{s/10 - 1}{s^2 + 10s + 400} + \frac{1000(1 - e^{-4s})}{s(s^2 + 10s + 400)}\right\} \\ &= \mathcal{L}^{-1}\left\{\frac{1}{10}\left(\frac{s - 10}{s^2 + 10s + 400}\right) + 1000\left(\frac{1/400}{s} - \frac{s/400 + 1/40}{s^2 + 10s + 400}\right)(1 - e^{-4s})\right\} \\ &= \mathcal{L}^{-1}\left\{\frac{1}{10}\left[\frac{(s + 5) - 15}{(s + 5)^2 + 375}\right] + \frac{5}{2}\left[\frac{1}{s} - \frac{(s + 5) + 5}{(s + 5)^2 + 375}\right](1 - e^{-4s})\right\} \\ &= \frac{e^{-5t}}{10}\left(\cos 5\sqrt{15}t - \frac{\sqrt{15}}{5}\sin 5\sqrt{15}t\right) + \frac{5}{2}\left[1 - e^{-5t}\left(\cos 5\sqrt{15}t + \frac{1}{\sqrt{15}}\sin 5\sqrt{15}t\right)\right] \\ &\quad - \frac{5}{2}\left\{1 - e^{-5(t-4)}\left[\cos 5\sqrt{15}(t-4) + \frac{1}{\sqrt{15}}\sin 5\sqrt{15}(t-4)\right]\right\}h(t-4) \\ &= \frac{5}{2} - \frac{12}{5}e^{-5t}\cos 5\sqrt{15}t - \frac{14\sqrt{15}}{75}e^{-5t}\sin 5\sqrt{15}t \\ &\quad - \frac{5}{2}\left\{1 - e^{5(4-t)}\left[\cos 5\sqrt{15}(t-4) + \frac{1}{\sqrt{15}}\sin 5\sqrt{15}(t-4)\right]\right\}h(t-4) \text{ m.} \end{aligned}$$

25. The initial-value problem for displacement of the mass from its equilibrium position is

$$\frac{1}{10}\frac{d^2x}{dt^2} + \frac{dx}{dt} + 40x = 100h(t-4), \quad x(0) = \frac{1}{10}, \quad x'(0) = -2.$$

We write the differential equation in the form

$$\frac{d^2x}{dt^2} + 10\frac{dx}{dt} + 400x = 1000h(t-4),$$

and take Laplace transforms,

$$\left[s^2X - \frac{s}{10} + 2\right] + 10\left[sX - \frac{1}{10}\right] + 400X = \frac{1000e^{-4s}}{s}.$$

We solve this for the transform $X(s)$,

$$X(s) = \frac{s/10 - 1}{s^2 + 10s + 400} + \frac{1000e^{-4s}}{s(s^2 + 10s + 400)}.$$

The inverse transform of this function is the solution of the initial-value problem

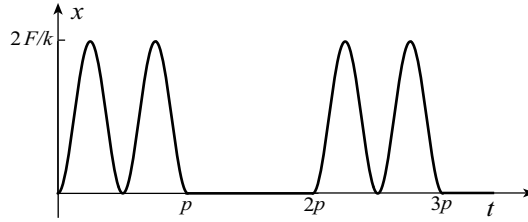
$$\begin{aligned} x(t) &= \mathcal{L}^{-1}\left\{\frac{s/10 - 1}{s^2 + 10s + 400} + \frac{1000e^{-4s}}{s(s^2 + 10s + 400)}\right\} \\ &= \mathcal{L}^{-1}\left\{\frac{1}{10}\left(\frac{s - 10}{s^2 + 10s + 400}\right) + 1000\left(\frac{1/400}{s} - \frac{s/400 + 1/40}{s^2 + 10s + 400}\right)e^{-4s}\right\} \\ &= \mathcal{L}^{-1}\left\{\frac{1}{10}\left[\frac{(s + 5) - 15}{(s + 5)^2 + 375}\right] + \frac{5}{2}\left[\frac{1}{s} - \frac{(s + 5) + 5}{(s + 5)^2 + 375}\right]e^{-4s}\right\} \\ &= \frac{e^{-5t}}{10}\left(\cos 5\sqrt{15}t - \frac{\sqrt{15}}{5}\sin 5\sqrt{15}t\right) \end{aligned}$$

$$\begin{aligned}
& + \frac{5}{2} \left\{ 1 - e^{-5(t-4)} \left[\cos 5\sqrt{15}(t-4) + \frac{1}{\sqrt{15}} \sin 5\sqrt{15}(t-4) \right] \right\} h(t-4) \\
= & \frac{e^{-5t}}{10} \left(\cos 5\sqrt{15}t - \frac{\sqrt{15}}{5} \sin 5\sqrt{15}t \right) \\
& + \frac{5}{2} \left\{ 1 - e^{5(4-t)} \left[\cos 5\sqrt{15}(t-4) + \frac{1}{\sqrt{15}} \sin 5\sqrt{15}(t-4) \right] \right\} h(t-4) \text{ m.}
\end{aligned}$$

26. When $p = 4\pi\sqrt{M/k}$, the displacements of Example 'Stationary' become

$$\begin{aligned}
x(t) &= \frac{F}{k} \sum_{n=0}^{\infty} (-1)^n \left[1 - \cos \frac{4\pi}{p}(t - pn) \right] h(t - pn) \\
&= \frac{F}{k} \sum_{n=0}^{\infty} (-1)^n \left[1 - \cos \frac{4\pi t}{p} \right] h(t - np) \\
&= \frac{F}{k} \left(1 - \cos \frac{4\pi t}{p} \right) \sum_{n=0}^{\infty} (-1)^n h(t - np).
\end{aligned}$$

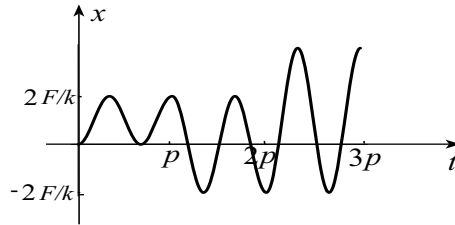
A graph of this function is shown below.



27. When $p = 3\pi\sqrt{M/k}$, the displacements of Example 'Stationary' become

$$\begin{aligned} x(t) &= \frac{F}{k} \sum_{n=0}^{\infty} (-1)^n \left[1 - \cos \frac{3\pi}{p}(t - pn) \right] h(t - pn) \\ &= \frac{F}{k} \sum_{n=0}^{\infty} (-1)^n \left[1 - (-1)^n \cos \frac{3\pi t}{p} \right] h(t - np) \\ &= \frac{F}{k} \sum_{n=0}^{\infty} \left[(-1)^n - \cos \frac{3\pi t}{p} \right] h(t - np). \end{aligned}$$

A graph of this function is shown below. There is resonance.



28. Since the force can be written algebraically as

$$20(t-1)[h(t-1) - h(t-2)] + 20h(t-2) = 20(t-1)h(t-1) + (40 - 20t)h(t-2),$$

the initial-value problem for displacements of the mass is

$$\frac{d^2x}{dt^2} + 16x = 20(t-1)h(t-1) + (40 - 20t)h(t-2), \quad x(0) = 0, \quad x'(0) = 0.$$

When we take Laplace transforms,

$$s^2X + 16X = 20e^{-s}\mathcal{L}\{t\} + e^{-2s}\mathcal{L}\{40 - 20(t+2)\} = \frac{20e^{-s}}{s^2} - \frac{20e^{-2s}}{s^2} = \frac{20(e^{-s} - e^{-2s})}{s^2}.$$

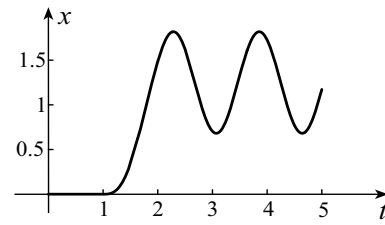
Solving for $X(s)$ gives

$$X(s) = \frac{20(e^{-s} - e^{-2s})}{s^2(s^2 + 16)} = 20(e^{-s} - e^{-2s}) \left(\frac{1/16}{s^2} - \frac{1/16}{s^2 + 16} \right).$$

Inverse transforms now yield

$$\begin{aligned} x(t) &= \frac{5}{4} \left[(t-1) - \frac{1}{4} \sin 4(t-1) \right] h(t-1) - \frac{5}{4} \left[(t-2) - \frac{1}{4} \sin 4(t-2) \right] h(t-2) \\ &= \frac{5}{16} [4(t-1) - \sin 4(t-1)] h(t-1) - \frac{5}{16} [4(t-2) - \sin 4(t-2)] h(t-2) \text{ m.} \end{aligned}$$

A plot of this function is shown to the right.



EXERCISES 6.5

1. The initial-value problem governing the number of grams $g(t)$ of salt in the tank is

$$\frac{dg}{dt} = \sum_{n=1}^{\infty} 500\delta(t - 60n) - \frac{5g}{10^6}, \quad g(0) = 5000.$$

When we take Laplace transforms,

$$sG - 5000 = \sum_{n=1}^{\infty} 500e^{-60ns} - \frac{5G}{10^6},$$

from which

$$G(s) = \frac{5000}{s + 5/10^6} + 500 \sum_{n=1}^{\infty} \frac{e^{-60ns}}{s + 5/10^6}.$$

Inverse transforms give

$$\begin{aligned} g(t) &= 5000e^{-5t/10^6} + 500 \sum_{n=1}^{\infty} e^{-5(t-60n)/10^6} h(t - 60n) \\ &= 500e^{-5t/10^6} \left[10 + \sum_{n=1}^{\infty} e^{3n/10^4} h(t - 60n) \right]. \end{aligned}$$

2. The initial-value problem governing the number of grams $g(t)$ of salt in the tank is

$$\frac{dg}{dt} = \frac{1}{10} + \sum_{n=1}^{\infty} 500\delta(t - 60n) - \frac{5g}{10^6}, \quad g(0) = 5000.$$

When we take Laplace transforms,

$$sG - 5000 = \frac{10}{s} + \sum_{n=1}^{\infty} 500e^{-60ns} - \frac{5G}{10^6},$$

from which

$$G(s) = \frac{10/s + 5000}{s + 5/10^6} + 500 \sum_{n=1}^{\infty} \frac{e^{-60ns}}{s + 5/10^6}.$$

Inverse transforms give

$$\begin{aligned} g(t) &= \mathcal{L}^{-1} \left\{ \frac{5000s + 10}{s(s + 5/10^6)} \right\} + 500 \sum_{n=1}^{\infty} e^{-5(t-60n)/10^6} h(t - 60n) \\ &= \mathcal{L}^{-1} \left\{ \frac{10^7/5}{s} - \frac{399(10^4)}{s + 5/10^6} \right\} + 500e^{-5t/10^6} \sum_{n=1}^{\infty} e^{3n/10^4} h(t - 60n) \\ &= 200\,000 - 1\,995\,000e^{-5t/10^6} + 500e^{-5t/10^6} \sum_{n=1}^{\infty} e^{3n/10^4} h(t - 60n). \end{aligned}$$

3. The initial-value problem for displacement of the mass from its equilibrium position is

$$2\frac{d^2x}{dt^2} + 512x = \delta(t), \quad x(0) = 0, \quad x'(0) = 0.$$

When we take Laplace transforms,

$$2[s^2X] + 512X = 1.$$

We solve this for the transform $X(s)$,

$$X(s) = \frac{1}{2s^2 + 512} = \frac{1}{2(s^2 + 256)}.$$

The inverse transform of this function is the solution of the initial-value problem

$$x(t) = \frac{1}{32} \sin 16t \text{ m.}$$

4. The initial-value problem for displacement of the mass from its equilibrium position is

$$2\frac{d^2x}{dt^2} + 80\frac{dx}{dt} + 512x = \delta(t), \quad x(0) = 0, \quad x'(0) = 0.$$

When we take Laplace transforms,

$$2[s^2X] + 80[sX] + 512X = 1.$$

We solve this for the transform $X(s)$,

$$X(s) = \frac{1}{2s^2 + 80s + 512} = \frac{1}{2(s^2 + 40s + 256)}.$$

The inverse transform of this function is the solution of the initial-value problem

$$\begin{aligned} x(t) &= \mathcal{L}^{-1} \left\{ \frac{1}{2(s^2 + 40s + 256)} \right\} = \frac{1}{2} \mathcal{L}^{-1} \left\{ \frac{1}{(s + 20)^2 - 144} \right\} \\ &= \frac{e^{-20t}}{2} \mathcal{L}^{-1} \left\{ \frac{1}{s^2 - 144} \right\} = \frac{e^{-20t}}{2} \mathcal{L}^{-1} \left\{ \frac{-1/24}{s + 12} + \frac{1/24}{s - 12} \right\} \\ &= \frac{e^{-20t}}{48} (-e^{-12t} + e^{12t}) = \frac{1}{48} (e^{-8t} - e^{-32t}) \text{ m.} \end{aligned}$$

5. The initial-value problem for displacement of the mass from its equilibrium position is

$$2\frac{d^2x}{dt^2} + 8\frac{dx}{dt} + 512x = \delta(t), \quad x(0) = 0, \quad x'(0) = 0.$$

When we take Laplace transforms,

$$2[s^2X] + 8[sX] + 512X = 1.$$

We solve this for the transform $X(s)$,

$$X(s) = \frac{1}{2s^2 + 8s + 512} = \frac{1}{2(s^2 + 4s + 256)}.$$

The inverse transform of this function is the solution of the initial-value problem

$$\begin{aligned} x(t) &= \mathcal{L}^{-1} \left\{ \frac{1}{2(s^2 + 4s + 256)} \right\} = \frac{1}{2} \mathcal{L}^{-1} \left\{ \frac{1}{(s + 2)^2 + 252} \right\} \\ &= \frac{e^{-2t}}{2} \mathcal{L}^{-1} \left\{ \frac{1}{s^2 + 252} \right\} = \frac{e^{-2t}}{12\sqrt{7}} \sin 6\sqrt{7}t \text{ m.} \end{aligned}$$

6. The initial-value problem for displacement of the mass from its equilibrium position is

$$2\frac{d^2x}{dt^2} + 512x = \delta(t - t_0), \quad x(0) = x_0, \quad x'(0) = 0.$$

When we take Laplace transforms,

$$2[s^2X - x_0s] + 512X = e^{-t_0s}.$$

We solve this for the transform $X(s)$,

$$X(s) = \frac{2x_0s}{2s^2 + 512} + \frac{e^{-t_0s}}{2s^2 + 512} = \frac{x_0s}{s^2 + 256} + \frac{e^{-t_0s}}{2(s^2 + 256)}.$$

The inverse transform of this function is the solution of the initial-value problem

$$x(t) = \mathcal{L}^{-1} \left\{ \frac{x_0s}{s^2 + 256} + \frac{e^{-t_0s}}{2(s^2 + 256)} \right\} = x_0 \cos 16t + \frac{1}{32} \sin 16(t - t_0) h(t - t_0) \text{ m.}$$

7. The initial-value problem for displacement of the mass from its equilibrium position is

$$2 \frac{d^2x}{dt^2} + 512x = \delta(t - t_0), \quad x(0) = 0, \quad x'(0) = v_0.$$

When we take Laplace transforms,

$$2[s^2X - v_0] + 512X = e^{-t_0s}.$$

We solve this for the transform $X(s)$,

$$X(s) = \frac{2v_0}{2s^2 + 512} + \frac{e^{-t_0s}}{2s^2 + 512} = \frac{v_0}{s^2 + 256} + \frac{e^{-t_0s}}{2(s^2 + 256)}.$$

The inverse transform of this function is the solution of the initial-value problem

$$x(t) = \mathcal{L}^{-1} \left\{ \frac{v_0}{s^2 + 256} + \frac{e^{-t_0s}}{2(s^2 + 256)} \right\} = \frac{v_0}{16} \sin 16t + \frac{1}{32} \sin 16(t - t_0) h(t - t_0) \text{ m.}$$

8. The initial-value problem for displacement of the mass from its equilibrium position is

$$2 \frac{d^2x}{dt^2} + 512x = \delta(t - t_0), \quad x(0) = x_0, \quad x'(0) = v_0.$$

When we take Laplace transforms,

$$2[s^2X - x_0s - v_0] + 512X = e^{-t_0s}.$$

We solve this for the transform $X(s)$,

$$X(s) = \frac{2x_0s + 2v_0}{2s^2 + 512} + \frac{e^{-t_0s}}{2s^2 + 512} = \frac{x_0s + v_0}{s^2 + 256} + \frac{e^{-t_0s}}{2(s^2 + 256)}.$$

The inverse transform of this function is the solution of the initial-value problem

$$x(t) = \mathcal{L}^{-1} \left\{ \frac{x_0s + v_0}{s^2 + 256} + \frac{e^{-t_0s}}{2(s^2 + 256)} \right\} = x_0 \cos 16t + \frac{v_0}{16} \sin 16t + \frac{1}{32} \sin 16(t - t_0) h(t - t_0) \text{ m.}$$

9. The initial-value problem for displacement of the mass from its equilibrium position is

$$\frac{d^2x}{dt^2} + 100x = \delta(t) + \delta(t - 1), \quad x(0) = 0, \quad x'(0) = 0.$$

When we take Laplace transforms,

$$[s^2X] + 100X = 1 + e^{-s}.$$

We solve this for the transform $X(s)$,

$$X(s) = \frac{1 + e^{-s}}{s^2 + 100}.$$

The inverse transform of this function is the solution of the initial-value problem

$$x(t) = \mathcal{L}^{-1} \left\{ \frac{1 + e^{-s}}{s^2 + 100} \right\} = \frac{1}{10} \sin 10t + \frac{1}{10} \sin 10(t - 1) h(t - 1) \text{ m.}$$

10. When we take Laplace transforms,

$$Ms^2X + kX = 1 \quad \implies \quad X(s) = \frac{1}{Ms^2 + k}.$$

The inverse transform is

$$x(t) = \frac{1}{M} \mathcal{L}^{-1} \left\{ \frac{1}{s^2 + k/M} \right\} = \frac{1}{\sqrt{kM}} \sin \sqrt{\frac{k}{M}} t.$$

This function satisfies the initial displacement condition $x(0) = 0$. The velocity of the mass is

$$v(t) = \frac{1}{\sqrt{kM}} \sqrt{\frac{k}{M}} \cos \sqrt{\frac{k}{M}} t = \frac{1}{M} \cos \sqrt{\frac{k}{M}} t.$$

The limit of this as $t \rightarrow 0$ is $1/M$. We have seen that an impulse force of size F causes a velocity change of F/M . To apply a unit impulse force at time $t = 0$ implies an instantaneous change of velocity $1/M$, and this is in conflict with the initial condition $x'(0) = 0$.

11. The initial-value problem for displacement of the mass from its equilibrium position is

$$\frac{d^2x}{dt^2} + 100x = \sum_{n=0}^{\infty} \delta(t - n), \quad x(0) = 0, \quad x'(0) = 0.$$

When we take Laplace transforms,

$$[s^2X] + 100X = \sum_{n=0}^{\infty} e^{-ns}.$$

We solve this for the transform $X(s)$,

$$X(s) = \sum_{n=0}^{\infty} \frac{e^{-ns}}{s^2 + 100}.$$

The inverse transform of this function is the solution of the initial-value problem

$$x(t) = \mathcal{L}^{-1} \left\{ \sum_{n=0}^{\infty} \frac{e^{-ns}}{s^2 + 100} \right\} = \frac{1}{10} \sum_{n=0}^{\infty} \sin 10(t - n) h(t - n) \text{ m.}$$

12. The initial-value problem for displacement of the mass from its equilibrium position is

$$\frac{d^2x}{dt^2} + 100x = \sum_{n=0}^{\infty} \delta(t - n\pi/5), \quad x(0) = 0, \quad x'(0) = 0.$$

When we take Laplace transforms,

$$[s^2X] + 100X = \sum_{n=0}^{\infty} e^{-n\pi s/5}.$$

We solve this for the transform $X(s)$,

$$X(s) = \sum_{n=0}^{\infty} \frac{e^{-n\pi s/5}}{s^2 + 100}.$$

The inverse transform of this function is the solution of the initial-value problem

$$x(t) = \mathcal{L}^{-1} \left\{ \sum_{n=0}^{\infty} \frac{e^{-n\pi s/5}}{s^2 + 100} \right\} = \frac{1}{10} \sum_{n=0}^{\infty} \sin 10(t - n\pi/5) h(t - n\pi/5) \text{ m.}$$

To show that resonance occurs we rewrite the displacement in the form

$$x(t) = \frac{1}{10} \sum_{n=0}^{\infty} \sin(10t - 2n\pi) h(t - n\pi/5) = \frac{1}{10} \sin 10t \sum_{n=0}^{\infty} h(t - n\pi/5).$$

This shows that $x(t)$ becomes unbounded for large t .

13. The initial-value problem for displacement of the mass is

$$M \frac{d^2x}{dt^2} + kx = F\delta(t - t_0), \quad x(0) = x_0, \quad x'(0) = v_0.$$

When we apply Laplace transforms,

$$M[s^2X - x_0s - v_0] + kX = Fe^{-t_0s},$$

from which

$$X(s) = \frac{M(x_0s + v_0)}{Ms^2 + k} + \frac{Fe^{-t_0s}}{Ms^2 + k} = \frac{x_0s + v_0}{s^2 + k/M} + \frac{F}{M} \left(\frac{e^{-t_0s}}{s^2 + k/M} \right).$$

The inverse Laplace transform is

$$\begin{aligned} x(t) &= x_0 \cos \sqrt{\frac{k}{M}}t + v_0 \sqrt{\frac{M}{k}} \sin \sqrt{\frac{k}{M}}t + \frac{F}{M} \sqrt{\frac{M}{k}} \sin \sqrt{\frac{k}{M}}(t - t_0) h(t - t_0) \\ &= x_0 \cos \sqrt{\frac{k}{M}}t + v_0 \sqrt{\frac{M}{k}} \sin \sqrt{\frac{k}{M}}t + \frac{F}{\sqrt{kM}} \sin \sqrt{\frac{k}{M}}(t - t_0) h(t - t_0). \end{aligned}$$

For the mass to come to an instantaneous stop at time $t = t_0$, we require

$$\begin{aligned} 0 &= \lim_{t \rightarrow t_0^+} x'(t) = \lim_{t \rightarrow t_0^+} \left[-x_0 \sqrt{\frac{k}{M}} \sin \sqrt{\frac{k}{M}}t + v_0 \cos \sqrt{\frac{k}{M}}t + \frac{F}{\sqrt{kM}} \sqrt{\frac{k}{M}} \cos \sqrt{\frac{k}{M}}(t - t_0) h(t - t_0) \right] \\ &= -x_0 \sqrt{\frac{k}{M}} \sin \sqrt{\frac{k}{M}}t_0 + v_0 \cos \sqrt{\frac{k}{M}}t_0 + \frac{F}{M}. \end{aligned}$$

We can solve this for F ,

$$F = x_0 \sqrt{kM} \sin \sqrt{\frac{k}{M}}t_0 - v_0 M \cos \sqrt{\frac{k}{M}}t_0.$$

14. (a) The initial-value problem for motion of the mass is

$$M \frac{d^2x}{dt^2} + kx = F\delta(t) - \mu Mg, \quad x(0) = 0, \quad x'(0) = 0.$$

When we take Laplace transforms, we get

$$Ms^2X + kX = F - \frac{\mu Mg}{s},$$

from which

$$X(s) = \frac{F}{Ms^2 + k} - \frac{\mu Mg}{s(Ms^2 + k)} = \frac{F}{M} \left(\frac{1}{s^2 + k/M} \right) - \mu Mg \left(\frac{1/k}{s} - \frac{s/k}{s^2 + k/M} \right).$$

Inverse transforms now give

$$x(t) = \frac{F}{\sqrt{kM}} \sin \sqrt{\frac{k}{M}}t - \frac{\mu Mg}{k} \left(1 - \cos \sqrt{\frac{k}{M}}t \right).$$

Since the velocity of the mass is

$$x'(t) = \frac{F}{M} \cos \sqrt{\frac{k}{M}}t - \mu g \sqrt{\frac{M}{k}} \sin \sqrt{\frac{k}{M}}t,$$

the initial velocity of imparted to the mass is $\lim_{t \rightarrow 0^+} x'(t) = \frac{F}{M}$.

(b) The mass comes to a stop when its velocity is zero. This will be the smallest positive solution of the equation

$$0 = \frac{F}{M} \cos \sqrt{\frac{k}{M}} t - \mu g \sqrt{\frac{M}{k}} \sin \sqrt{\frac{k}{M}} t \implies \tan \sqrt{\frac{k}{M}} t = \frac{F/M}{\mu g \sqrt{M/k}} = \frac{F\sqrt{k}}{\mu g M^{3/2}}.$$

Solutions of this equation are

$$t = \sqrt{\frac{M}{k}} \left[\text{Tan}^{-1} \left(\frac{F\sqrt{k}}{\mu g M^{3/2}} \right) + n\pi \right],$$

where n is an integer. The smallest positive solution is $t_0 = \sqrt{\frac{M}{k}} \text{Tan}^{-1} \left(\frac{F\sqrt{k}}{\mu g M^{3/2}} \right)$.

(c) The position of the mass at this time is

$$x(t_0) = \frac{F}{\sqrt{kM}} \sin \sqrt{\frac{k}{M}} t_0 - \frac{\mu Mg}{k} \left(1 - \cos \sqrt{\frac{k}{M}} t_0 \right).$$

Since $\tan \sqrt{\frac{k}{M}} t_0 = \frac{F\sqrt{k}}{\mu g M^{3/2}}$, it follows that

$$\sin \sqrt{\frac{k}{M}} t_0 = \frac{F\sqrt{k}}{\sqrt{F^2 k + \mu^2 g^2 M^3}}, \quad \cos \sqrt{\frac{k}{M}} t_0 = \frac{\mu g M^{3/2}}{\sqrt{F^2 k + \mu^2 g^2 M^3}}.$$

With these,

$$\begin{aligned} x(t_0) &= \frac{F}{\sqrt{kM}} \frac{F\sqrt{k}}{\sqrt{F^2 k + \mu^2 g^2 M^3}} - \frac{\mu Mg}{k} \left(1 - \frac{\mu g M^{3/2}}{\sqrt{F^2 k + \mu^2 g^2 M^3}} \right) \\ &= -\frac{\mu Mg}{k} + \frac{1}{\sqrt{F^2 k + \mu^2 g^2 M^3}} \left(\frac{F^2}{\sqrt{M}} + \frac{\mu^2 g^2 M^{5/2}}{k} \right) \\ &= -\frac{\mu Mg}{k} + \frac{1}{k\sqrt{M}\sqrt{F^2 k + \mu^2 g^2 M^3}} (F^2 k + \mu^2 g^2 M^3) = -\frac{\mu Mg}{k} + \frac{\sqrt{F^2 k + \mu^2 g^2 M^3}}{k\sqrt{M}}. \end{aligned}$$

The spring force at this displacement is

$$x(t_0) = k \left(-\frac{\mu Mg}{k} + \frac{\sqrt{F^2 k + \mu^2 g^2 M^3}}{k\sqrt{M}} \right) = -\mu Mg + \frac{\sqrt{F^2 k + \mu^2 g^2 M^3}}{\sqrt{M}}.$$

Since the force of static friction is $\mu_s Mg$, the mass moves to the left if

$$\begin{aligned} -\mu Mg + \frac{\sqrt{F^2 k + \mu^2 g^2 M^3}}{\sqrt{M}} &> \mu_s Mg \\ F^2 k + \mu^2 g^2 M^3 &> M^3 g^2 (\mu_s + \mu)^2 \\ F^2 k &> M^3 g^2 (\mu_s^2 + 2\mu\mu_s) \\ F &> \sqrt{\frac{g^2 M^3 \mu_s (\mu_s + 2\mu)}{k}}. \end{aligned}$$

15. If we write $x(t)$ in the form

$$\begin{aligned} x(t) &= \frac{F}{ka} \left[\cos \sqrt{\frac{k}{M}} (t - t_0) \cos \sqrt{\frac{k}{M}} a + \sin \sqrt{\frac{k}{M}} (t - t_0) \sin \sqrt{\frac{k}{M}} a - \cos \sqrt{\frac{k}{M}} (t - t_0) \right] \\ &= \frac{F}{ka} \left[\left(\cos \sqrt{\frac{k}{M}} a - 1 \right) \cos \sqrt{\frac{k}{M}} (t - t_0) + \sin \sqrt{\frac{k}{M}} a \sin \sqrt{\frac{k}{M}} (t - t_0) \right], \end{aligned}$$

the amplitude of the simple harmonic motion is

$$\begin{aligned} A &= \frac{F}{ka} \sqrt{\left(\cos \sqrt{\frac{k}{M}} a - 1\right)^2 + \sin^2 \sqrt{\frac{k}{M}} a} = \frac{F}{ka} \sqrt{2 - 2 \cos \sqrt{\frac{k}{M}} a} \\ &= \frac{\sqrt{2}F}{ka} \sqrt{1 - \left(1 - 2 \sin^2 \sqrt{\frac{k}{M}} \frac{a}{2}\right)} = \frac{2F}{ka} \left| \sin \sqrt{\frac{k}{M}} \frac{a}{2} \right|. \end{aligned}$$

It is a question of whether $2 \left| \sin \sqrt{\frac{k}{M}} \frac{a}{2} \right|$ can be less than one for some values of a and greater than

one for other values of a . This is certainly possible. If a is such that $\sqrt{\frac{k}{M}} \frac{a}{2}$ is close to $\pi/2$, then

$2 \left| \sin \sqrt{\frac{k}{M}} \frac{a}{2} \right|$ will be greater than one, whereas if $\sqrt{\frac{k}{M}} \frac{a}{2}$ is close to zero, then $2 \left| \sin \sqrt{\frac{k}{M}} \frac{a}{2} \right|$ will be less than one.

16. (a) With $v = 20$ km/hr or $50/9$ m/s, the initial-value problem for displacement of the front end of the car is

$$200 \frac{d^2 y}{dt^2} + 1000y = f(50t/9), \quad y(0) = 0, \quad y'(0) = 0,$$

where

$$\begin{aligned} f(50t/9) &= \begin{cases} \frac{3}{5} \left(\frac{50t}{9}\right) \left(1 - \frac{50t}{9}\right), & 0 < t < 9/50 \\ 0, & t > 9/50 \end{cases} \\ &= \frac{10t}{27} (9 - 50t) \left[h(t) - h\left(t - \frac{9}{50}\right) \right]. \end{aligned}$$

When we take Laplace transforms,

$$\begin{aligned} 200s^2 Y + 1000Y &= \frac{10}{27} \left[\left(\frac{9}{s^2} - \frac{100}{s^3} \right) - e^{-9s/50} \mathcal{L} \left\{ \left(t + \frac{9}{50} \right) \left[9 - 50 \left(t + \frac{9}{50} \right) \right] \right\} \right] \\ &= \frac{10}{27} \left[\left(\frac{9}{s^2} - \frac{100}{s^3} \right) - e^{-9s/50} \mathcal{L} \{ -9t - 50t^2 \} \right] \\ &= \frac{10}{27} \left[\left(\frac{9}{s^2} - \frac{100}{s^3} \right) + e^{-9s/50} \left(\frac{9}{s^2} + \frac{100}{s^3} \right) \right]. \end{aligned}$$

We now solve for $Y(s)$,

$$\begin{aligned} Y(s) &= \frac{10}{27(200s^2 + 1000)} \left[\left(\frac{9}{s^2} - \frac{100}{s^3} \right) + e^{-9s/50} \left(\frac{9}{s^2} + \frac{100}{s^3} \right) \right] \\ &= \frac{1}{540(s^2 + 5)} \left[\left(\frac{9s - 100}{s^3} \right) + e^{-9s/50} \left(\frac{9s + 100}{s^3} \right) \right]. \end{aligned}$$

Partial fractions give

$$Y(s) = \left[\frac{4}{s} + \frac{9}{5s^2} - \frac{20}{s^3} - \frac{20s + 9}{5(s^2 + 5)} \right] + e^{-9s/50} \left[-\frac{4}{s} + \frac{9}{5s^2} + \frac{20}{s^3} + \frac{20s - 9}{5(s^2 + 5)} \right].$$

Inverse transforms now yield

$$\begin{aligned} y(t) &= 4 + \frac{9t}{5} - 10t^2 - 4 \cos \sqrt{5}t - \frac{9\sqrt{5}}{25} \sin \sqrt{5}t + \left[-4 + \frac{9}{5} \left(t - \frac{9}{50} \right) + 10 \left(t - \frac{9}{50} \right)^2 \right. \\ &\quad \left. + 4 \cos \sqrt{5} \left(t - \frac{9}{50} \right) - \frac{9\sqrt{5}}{25} \sin \sqrt{5} \left(t - \frac{9}{50} \right) \right] h \left(t - \frac{9}{50} \right). \end{aligned}$$

(b) After the speed bump (when $t > 9/50$), the displacement of the car is

$$\begin{aligned}
 y(t) &= 4 + \frac{9t}{5} - 10t^2 - 4 \cos \sqrt{5}t - \frac{9\sqrt{5}}{25} \sin \sqrt{5}t + \left[-4 + \frac{9}{5} \left(t - \frac{9}{50} \right) + 10 \left(t - \frac{9}{50} \right)^2 \right. \\
 &\quad \left. + 4 \cos \sqrt{5} \left(t - \frac{9}{50} \right) - \frac{9\sqrt{5}}{25} \sin \sqrt{5} \left(t - \frac{9}{50} \right) \right] \\
 &= -4 \cos \sqrt{5}t - \frac{9\sqrt{5}}{25} \sin \sqrt{5}t + 4 \cos \sqrt{5} \left(t - \frac{9}{50} \right) - \frac{9\sqrt{5}}{25} \sin \sqrt{5} \left(t - \frac{9}{50} \right) \\
 &= -4 \cos \sqrt{5}t - \frac{9\sqrt{5}}{25} \sin \sqrt{5}t + 4 \left(\cos \sqrt{5}t \cos \frac{9\sqrt{5}}{50} + \sin \sqrt{5}t \sin \frac{9\sqrt{5}}{50} \right) \\
 &\quad - \frac{9\sqrt{5}}{25} \left(\sin \sqrt{5}t \cos \frac{9\sqrt{5}}{50} - \cos \sqrt{5}t \sin \frac{9\sqrt{5}}{50} \right) \\
 &= \left(-4 + 4 \cos \frac{9\sqrt{5}}{50} + \frac{9\sqrt{5}}{25} \sin \frac{9\sqrt{5}}{50} \right) \cos \sqrt{5}t \\
 &\quad + \left(-\frac{9\sqrt{5}}{25} + 4 \sin \frac{9\sqrt{5}}{50} - \frac{9\sqrt{5}}{25} \cos \frac{9\sqrt{5}}{50} \right) \sin \sqrt{5}t.
 \end{aligned}$$

This is simple harmonic motion with amplitude

$$\sqrt{\left(-4 + 4 \cos \frac{9\sqrt{5}}{50} + \frac{9\sqrt{5}}{25} \sin \frac{9\sqrt{5}}{50} \right)^2 + \left(-\frac{9\sqrt{5}}{25} + 4 \sin \frac{9\sqrt{5}}{50} - \frac{9\sqrt{5}}{25} \cos \frac{9\sqrt{5}}{50} \right)^2} = 0.022.$$

The amplitude is 2.2 centimetres.

17. When we take Laplace transforms of the differential equation, and use the initial conditions,

$$(s^2Y - s + 2) - 3(sY - 1) - 4Y = F(s) \quad \Longrightarrow \quad Y(s) = \frac{s - 5}{s^2 - 3s - 4} + \frac{F(s)}{s^2 - 3s - 4}.$$

The transfer function is $H(s) = 1/(s^2 - 3s - 4)$, and the unit impulse response function is its inverse,

$$h(t) = \mathcal{L}^{-1} \left\{ \frac{1}{(s-4)(s+1)} \right\} = \mathcal{L}^{-1} \left\{ \frac{1/5}{s-4} - \frac{1/5}{s+1} \right\} = \frac{1}{5}(e^{4t} - e^{-t}).$$

A general solution of the associated homogeneous problem is

$$y_h(t) = \mathcal{L}^{-1} \left\{ \frac{s-5}{(s-4)(s+1)} \right\} = \mathcal{L}^{-1} \left\{ \frac{-1/5}{s-4} + \frac{6/5}{s+1} \right\} = \frac{1}{5}(6e^{-t} - e^{4t}).$$

Equation 'Soln with impulse response' gives the solution of the initial-value problem in the form

$$y(t) = \frac{1}{5}(6e^{-t} - e^{4t}) + \frac{1}{5} \int_0^t f(u)[6e^{-(t-u)} + e^{4(t-u)}] du.$$

18. When we take Laplace transforms of the differential equation, and use the initial conditions,

$$(s^2Y - As - B) + 2(sY - A) + 3Y = F(s) \quad \Longrightarrow \quad Y(s) = \frac{As + (B + 2A)}{s^2 + 2s + 3} + \frac{F(s)}{s^2 + 2s + 3}.$$

The transfer function is $H(s) = 1/(s^2 + 2s + 3)$, and the unit impulse response function is its inverse,

$$h(t) = \mathcal{L}^{-1} \left\{ \frac{1}{(s+1)^2 + 2} \right\} = e^{-t} \mathcal{L}^{-1} \left\{ \frac{1}{s^2 + 2} \right\} = \frac{1}{\sqrt{2}} e^{-t} \sin \sqrt{2}t.$$

A general solution of the associated homogeneous problem is

$$\begin{aligned} y_h(t) &= \mathcal{L}^{-1} \left\{ \frac{As + (B + 2A)}{s^2 + 2s + 3} \right\} = \mathcal{L}^{-1} \left\{ \frac{A(s + 1) + (A + B)}{(s + 1)^2 + 2} \right\} \\ &= e^{-t} \mathcal{L}^{-1} \left\{ \frac{As + (A + B)}{s^2 + 2} \right\} = e^{-t} \left[A \cos \sqrt{2}t + \left(\frac{A + B}{\sqrt{2}} \right) \sin \sqrt{2}t \right]. \end{aligned}$$

Equation 'Soln with impulse response' gives the solution of the initial-value problem in the form

$$y(t) = e^{-t} \left[A \cos \sqrt{2}t + \left(\frac{A + B}{\sqrt{2}} \right) \sin \sqrt{2}t \right] + \frac{1}{\sqrt{2}} \int_0^t f(u) e^{-(t-u)} \sin \sqrt{2}(t-u) du.$$

19. The unit impulse response function $\hbar(t)$ is the inverse transform of $1/P(s)$ which we would find by factoring $P(s)$. A real zero $s = a$ of $P(s)$ leads to a term e^{at} in $\hbar(t)$. It is bounded if, and only if, $a \leq 0$. Complex conjugate zeros $a \pm bi$ lead to terms $e^{at} \cos bt$ and $e^{at} \sin at$. These are bounded if, and only if, $a \leq 0$.
20. The unit impulse response function $\hbar(t)$ is the inverse transform of $1/P(s)$ which we would find by factoring $P(s)$. A real zero $s = a$ of $P(s)$ leads to a term e^{at} in $\hbar(t)$. It approaches zero if, and only if, $a < 0$. Complex conjugate zeros $a \pm bi$ lead to terms $e^{at} \cos bt$ and $e^{at} \sin at$. These approach zero if, and only if, $a < 0$.