CHAPTER 5 APPLICATIONS OF LINEAR DIFFERENTIAL EQUATIONS

In Chapter 3 we saw that a single differential equation can model many different situations. The linear second-order differential equation, to which we paid so much attention in Chapter 4, represents so many applications, it is undoubedly the most widely used differential equation in applied mathematics, the physical sciences, and engineering. We concentrate on one of these applications in the first three sections of this chapter, vibrations of masses on the ends of springs. Simplistic as this situation may seem, it serves as a beginning model for many complicated vibrating systems. In addition, whatever results we discover about vibrating mass-spring systems can be interpreted in other physical systems described by linear second-order differential equations.

5.1 Vibrating Mass-Spring Systems

Consider the situation in Figure 5.1 of a spring attached to a solid wall on one end and a mass on the other. If the mass is somehow set into horizontal motion along the axis of the spring it will continue to do so for some time. The nature of the motion depends on a number of factors such as the tightness of the spring, the amount of mass, whether there is friction between the mass and the surface along which it slides, whether there is friction between the mass and the medium in which it slides, and whether there are any other forces acting on the mass. In this and the next two sections we model this situation with linear second-order differential equations.

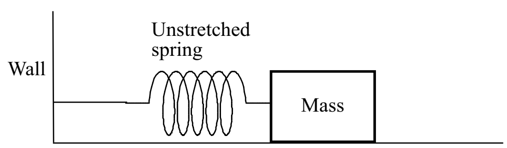


Figure 5.1

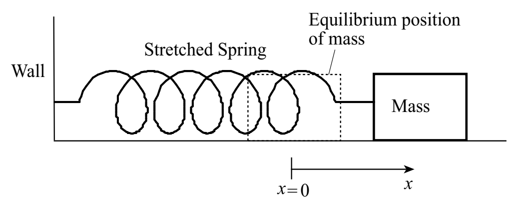


Figure 5.2

Our objective is to predict the position of the mass at any given time, knowing the forces acting on the mass, and how motion is initiated. We begin by establishing a means by which to identify the position of the mass. Most convenient is to let x represent the position of the mass relative to the position that it would occupy were the spring unstretched and uncompressed, usually called the **equilibrium position** (Figure 5.2). We shall then look for x as a function of time t, taking $t = 0$ at the instant that motion is initiated. To determine the differential equation describing oscillations of the mass, we analyze the forces acting on the mass when it is at position x. First there is the spring. Hooke's Law states that when a spring is stretched, the force exerted by the spring in an attempt to restore itself to an unstretched position is proportional to the amount of stretch in the spring. Since x not only identifies the position of the mass, but also represents the stretch in the spring, it follows that the restoring force exerted by the spring on the mass at position x is $-kx$, where $k > 0$ is the constant of proportionality, called the **spring constant**. The negative sign indicates that the force is to the left when x is positive and the spring is stretched. In a compressed situation, the spring force should be positive (to the right). This is indeed the case, because with compression,

x is negative, and therefore $-kx$ is positive.

In many vibration problems, there is a **damping force**, a force opposing motion that has magnitude directly proportional to the velocity of the mass. It might be a result of air friction with the mass, or it might be due to a mechanical device like a shock absorber on a car, or a combination of such forces. Damping forces are modelled by what is called a **dashpot**; it is shown in (Figure 5.3). Because damping forces are proportional to velocity, and velocity is given by dx/dt, they can be represented in the form $-\beta(dx/dt)$, where $\beta > 0$ is a constant. The negative sign accounts for the fact that damping forces oppose motion; they are in the opposite direction to velocity.

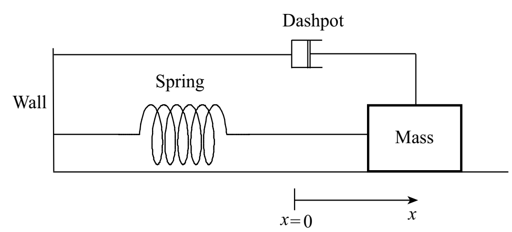


Figure 5.3

There could be other forces that act on the mass; they could depend on both the position of the mass and time. In the event that they depend only on time, and this is the only case that we consider in this chapter, we denote them by $F(t)$. The total force acting on the mass is therefore $-kx - \beta(dx/dt) + F(t)$. According to Newton's second law, the acceleration a of the mass due to this force is equal to the force divided by the mass M; that is $a = F/M$, or $F = Ma$. Since acceleration is the second derivative of the displacement or position function, d^2x/dt^2, we can write that

$$-kx - \beta\frac{dx}{dt} + F(t) = M\frac{d^2x}{dt^2}. \tag{5.1}$$

When this equation is rearranged into the form,

$$M\frac{d^2x}{dt^2} + \beta\frac{dx}{dt} + kx = F(t), \tag{5.2}$$

we have a linear, second-order differential equation for the position function $x(t)$. The equation is homogeneous or nonhomogeneous depending on whether forces other than the spring and damping forces act on the mass. In this section and the next, we consider situations in which these are the only two forces acting on the mass; the presence of periodic forces in Section 5.3 leads to the extremely important concept of resonance.

Before considering specific situations, we show that when masses are suspended vertically from springs, their motion is also governed by equation 5.2. To describe the position of the mass M in Figure 5.4 as a function of time t, we choose a vertical coordinate system. There are two natural places to choose the origin. One is to choose $y = 0$ at the position of M when the spring is unstretched. Suppose we do this and choose y as positive upward. When M is a distance y away from the origin, the restoring force of the spring is $-ky$. In addition, if $g = 9.81$ is the acceleration due to gravity, then the force of gravity on M is $-Mg$. In the presence of damping forces or a dashpot, there is a force of the form $-\beta(dy/dt)$, where β is a positive constant. If $F(t)$ represents all other forces acting on M, then the total force on M is $-ky - Mg - \beta(dy/dt) + F(t)$, and Newton's second law for the acceleration of M gives

$$-ky - Mg - \beta\frac{dy}{dt} + F(t) = M\frac{d^2y}{dt^2}.$$

Consequently, the differential equation that determines the position $y(t)$ of M relative to the unstretched position of the spring is

$$M\frac{d^2y}{dt^2} + \beta\frac{dy}{dt} + ky = -Mg + F(t). \tag{5.3}$$

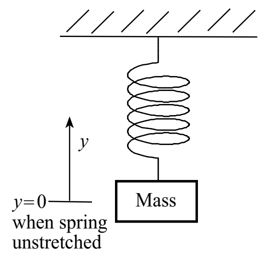


Figure 5.4

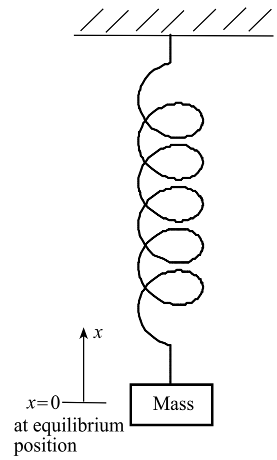


Figure 5.5

The alternative possibility for describing vertical oscillations is to attach M to the spring and slowly lower M until it reaches an equilibrium position. At this position, the restoring force of the spring is exactly equal to the force of gravity on the mass, and the mass, left by itself, will remain motionless. If s is the amount of stretch in the spring at equilibrium, and g is the acceleration due to gravity, then at equilibrium

$$ks - Mg = 0, \qquad \text{where } s > 0 \text{ and } g > 0. \tag{5.4}$$

Suppose we take the equilibrium position as $x = 0$ and x as positive upward (Figure 5.5). When M is a distance x away from its equilibrium position, the restoring force of the spring on M is $k(s-x)$. The force of gravity remains as $-Mg$, and that of the damping force is $-\beta(dx/dt)$. If $F(t)$ accounts for any other forces acting on M, Newton's second law implies that

$$M\frac{d^2x}{dt^2} = k(s-x) - Mg - \beta\frac{dx}{dt} + F(t),$$

or,

$$M\frac{d^2x}{dt^2} + \beta\frac{dx}{dt} + kx = -Mg + ks + F(t).$$

But according to equation 5.4, $ks - Mg = 0$, and hence

$$M\frac{d^2x}{dt^2} + \beta\frac{dx}{dt} + kx = F(t). \tag{5.5}$$

This is the differential equation describing the displacement $x(t)$ of M relative to the equilibrium position of M.

Equations 5.3 and 5.5 are both linear second-order differential equations with constant coefficients. The advantage of equation 5.5 is that nonhomogeneity $-Mg$ is absent, and this is simply due to a convenient choice of dependent variable (x as opposed to y). Physically, there are two parts to the spring force $k(s-x)$; a part ks and a part $-kx$. Gravity is always acting on M, and that part ks of the spring force is counteracting gravity in an attempt to restore the spring to its unstretched position. Because these forces always cancel, we might just as well eliminate both of them from our discussion. This leaves $-kx$, and we therefore interpret $-kx$ as the spring force attempting to restore the mass to its equilibrium position.

If we choose equation 5.5 to describe vertical oscillations (and this equation is usually chosen over 5.3), we must remember three things: x is measured from equilibrium, $-kx$ is the spring force attempting to restore M to its equilibrium position, and gravity has been taken into account.

Equation 5.5 for vertical oscillations and equation 5.2 for horizontal oscillations are identical; we have the same differential equation describing either type of oscillation. In both cases, x measures the distance of the mass from its equilibrium position. In the horizontal case, this is from the position of the mass when the spring is unstretched; in the vertical case, this is from the position of the mass when it hangs motionless at the end of the spring.

There are three basic ways to initiate motion. First, we can move the mass away from its equilibrium position and then release it, giving it an initial displacement but no initial velocity. Secondly, we can strike the mass at the equilibrium position, imparting an initial velocity but no initial displacement. And finally, we can give the mass both an initial displacement and an initial velocity. Each of these methods adds two initial conditions to the differential equation.

Example 5.1 A 2-kilogram mass is suspended vertically from a spring with constant 32 newtons per metre. The mass is raised 10 centimetres above its equilibrium position and then released. If damping is ignored, find the position of the mass as a function of time.

Solution If we choose $x = 0$ at the equilibrium position of the mass and x positive upward, differential equation 5.5 for the motion $x(t)$ of the mass becomes

$$2\frac{d^2x}{dt^2} + 32x = 0, \qquad \text{or,} \qquad \frac{d^2x}{dt^2} + 16x = 0,$$

along with the initial conditions $x(0) = 1/10$, $x'(0) = 0$. The auxiliary equation $m^2 + 16 = 0$ has solutions $m = \pm 4i$. Consequently,

$$x(t) = C_1 \cos 4t + C_2 \sin 4t.$$

The initial conditions require

$$1/10 = C_1, \quad 0 = 4C_2.$$

Thus,

$$x(t) = \frac{1}{10}\cos 4t \text{ m}.$$

A graph of this function (Figure 5.6) illustrates that the mass oscillates about its equilibrium position forever. This is a direct result of the fact that damping has been ignored. The mass oscillates up and down from a position 10 cm above the equilibrium position to a position 10 cm below the equilibrium position. We call 10 cm the **amplitude** of the oscillations. It takes $2\pi/4 = \pi/2$ seconds to complete one full oscillation, and we call this the **period** of the oscillations. The **frequency** of the oscillations is the number of oscillations that take place each second and this is the reciprocal of the period, namely $2/\pi$ Hz (hertz). Oscillations of this kind are called **simple harmonic motion**.•

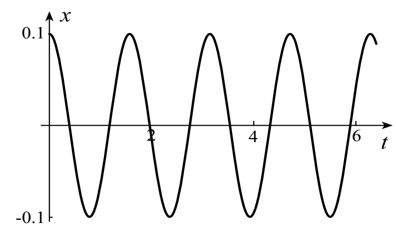


Figure 5.6

The spring in this example might be called "loose". We can see this from equation 5.4. Substitution of $M = 2$, $k = 32$, and $g = 9.81$ gives $s = 0.61$ metres; that is, with a 2 kilogram mass suspended at rest from the spring there is a stretch of 61 centimetres. The period of oscillations $\pi/2$ is quite long and the frequency of oscillations is small $2/\pi$. Contrast this with what might be called a tight spring in the following example.

Example 5.2 The 2-kilogram mass in Example 5.1 is suspended vertically from a spring with constant 3200 newtons per metre. The mass is raised 10 centimetres above its equilibrium position and given an initial velocity of 2 metres per second downward. If damping is ignored, find the position of the mass as a function of time.

Solution The differential equation governing motion is

$$2\frac{d^2x}{dt^2} + 3200x = 0, \qquad \text{or,} \qquad \frac{d^2x}{dt^2} + 1600x = 0,$$

along with the initial conditions $x(0) = 1/10$, $x'(0) = -2$. The auxiliary equation $m^2 + 1600 = 0$ has solutions $m = \pm 40i$. Consequently,

$$x(t) = C_1 \cos 40t + C_2 \sin 40t.$$

The initial conditions require

$$1/10 = C_1, \quad -2 = 40C_2.$$

Thus,

$$x(t) = \frac{1}{10}\cos 40t - \frac{1}{20}\sin 40t \text{ m}.$$

It is more convenient to express this function in the form $A\sin(40t + \phi)$. To find A and ϕ, we set

$$\frac{1}{10}\cos 40t - \frac{1}{20}\sin 40t = A\sin(40t + \phi) = A[\sin 40t \cos\phi + \cos 40t \sin\phi].$$

Because $\sin 40t$ and $\cos 40t$ are linearly independent functions we equate coefficients to obtain

$$\frac{1}{10} = A\sin\phi, \qquad \frac{-1}{20} = A\cos\phi.$$

When these are squared and added,

$$\frac{1}{100} + \frac{1}{400} = A^2 \qquad \Longrightarrow \qquad A = \frac{\sqrt{5}}{20}.$$

It now follows that ϕ must satisfy the equations

$$\frac{1}{10} = \frac{\sqrt{5}}{20}\sin\phi, \qquad \frac{-1}{20} = \frac{\sqrt{5}}{20}\cos\phi.$$

The smallest positive angle satisfying these is $\phi = 2.03$ radians. The position function of the mass is therefore

$$x(t) = \frac{\sqrt{5}}{20}\sin(40t + 2.03)\ \text{m},$$

a graph of which is shown in Figure 5.7. The amplitude $\sqrt{5}/20$ of the oscillations is slightly larger than that in Example 5.1 due to the fact that the mass was given not only an initial displacement of 10 cm, but also an initial velocity. The spring, with constant $k = 3200$, is one hundred times as tight as that in Example 5.1. The result is a period $\pi/20$ s for the oscillations, one-tenth that in Example 5.1, and ten times as many oscillations per second (frequency is $20/\pi$ Hz).•

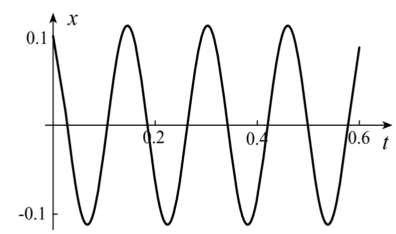


Figure 5.7

EXERCISES 5.1

1. A 1-kilogram mass is suspended vertically from a spring with constant 16 newtons per metre. The mass is pulled 10 centimetres below its equilibrium position and then released. Find the position of the mass, relative to its equilibrium position, at any time if damping is ignored.

2. A 100-gram mass is attached to a spring with constant 100 newtons per metre as in Figure 5.2. The mass is pulled 5 centimetres to the right and released. Find the position of the mass if damping, and friction over the sliding surface, are ignored. Sketch a graph of the position function identifying the amplitude, period, and frequency of the oscillations.

3. Repeat Exercise 2 if motion is initiated by striking the mass, at equilibrium, so as to impart a velocity of 3 metres per second to the left.

4. Repeat Exercise 2 if motion is initiated by pulling the mass 5 centimetres to the right and giving it an initial velocity 3 metres per second to the left.

5. Repeat Exercise 2 if motion is initiated by pulling the mass 5 centimetres to the left and giving it an initial velocity 3 metres per second to the left.

6. (a) A 2-kilogram mass is suspended from a spring with constant 1000 newtons per metre. If the mass is pulled 3 centimetres below its equilibrium position and given a downward velocity of 2 metres per second, find its position thereafter. Sketch a graph of the position function identifying the amplitude, period, and frequency of the oscillations.
   (b) Do the initial displacement and velocity affect the amplitude, period, and/or frequency?

7. If the mass in Exercise 6 is quadrupled, how does this affect the period and frequency of the oscillations?

8. If the spring constant in Exercise 6 is quadrupled, how does this affect the period and frequency of the oscillations?

9. When a 2-kilogram mass is set into vertical vibrations on the end of a spring, 3 full oscillations occur each second. What is the spring constant if there is no damping?

10. A mass M is suspended from a spring with constant k. Oscillations are initiated by giving the mass a displacement x_0 and velocity v_0. Show that the position of the mass relative to its equilibrium position, when damping is ignored, can be expressed in the form

$$x(t) = A \sin\left(\sqrt{k/M}t + \phi\right),$$

where the amplitude is $A = \sqrt{x_0^2 + Mv_0^2/k}$, and ϕ satisfies

$$\sin\phi = \frac{x_0}{A}, \qquad \cos\phi = \frac{\sqrt{M/k}v_0}{A}.$$

11. Use the result of Exercise 10 to show that when the mass on the end of a spring is doubled, the period increases by a factor of $\sqrt{2}$ and the frequency decreases by a factor of $1/\sqrt{2}$.

12. Show the following for oscillations of a mass on the end of a spring when damping is ignored:
(a) Maximum velocity occurs when the mass passes through its equilibrium position. What is the acceleration at this instant?
(b) Maximum acceleration occurs when the mass is at its maximum distance from equilibrium. What is the velocity there?

13. When a spring is suspended vertically, its own weight causes it to stretch. Would this have any effect on our analysis of motion of a mass suspended from the spring?

14. A 100-gram mass is suspended vertically from a spring with constant 40 newtons per metre. The mass is pulled 2 centimetres below its equilibrium position and given an upward velocity of 10 metres per second. Determine:
(a) the position of the mass as a function of time
(b) the amplitude, period, and frequency of the oscillations
(c) all times when the mass has velocity zero
(d) all times when the mass passes through the equilibrium position
(e) all times when the mass has velocity 2 metres per second
(f) all times when the mass is 1 centimetre above the equilibrium position
(g) whether the mass ever has velocity 12 metres per second
(h) the second time the mass is at a maximum height above the equilibrium position.

15. At time $t = 0$, a mass M is attached to the end of a hanging spring with constant k, and then released. Assuming that damping is negligible, find the subsequent displacement of the mass as a function of time.

16. A container of water has mass M kilograms of which m kilograms is water. At time $t = 0$, the container is attached to a spring with constant k newtons per metre, and released. A hole in the bottom of the container allows water to run out at the constant rate of r kilograms per second. If air resistance proportional to velocity acts on the container, set up an initial-value problem for the position of the container while water remains in the container. Do so with the coordinate systems of (a) Figure 5.4, where $y = 0$ is the unstretched position of the spring, and

(b) Figure 5.5, where $x = 0$ corresponds to the equilibrium position for a full container. Can you solve these problems?

17. (a) A cube L metres on each side and with mass M kilograms floats half submerged in water. If it is pushed down slightly and then released, oscillations take place. Use Archimedes' principle to find the differential equation governing these oscillations. Assume no damping forces due to the viscosity of the water.
(b) What is the frequency of the oscillations?

18. A cylindrical buoy 20 centimetres in diameter floats partially submerged with its axis vertical. When it is depressed slightly and released, its oscillations have a period equal to 4 seconds. What is the mass of the buoy?

19. A sphere of radius R floats half submerged in water. It is set into vibration by pushing it down slightly and then releasing it. If y denotes the instantaneous distance of its centre below the surface, show that

$$\frac{d^2y}{dt^2} = \frac{-3g}{2R^3}\left(R^2y - \frac{y^3}{3}\right),$$

where g is the acceleration due to gravity. Is this a linear differential equation?

5.2 Vibrating Mass-Spring Systems With Damping

Vibrating mass-spring systems without damping are unrealistic. All vibrations are subject to damping of some sort, and depending on the magnitude of the damping, oscillations either gradually die out, or are completely expunged. Differential equation 5.5 describes the motion of a mass on the end of a spring in the presence of a damping force with damping constant β. When no other forces act on the mass, besides the spring, and gravity for vertical oscillations, the differential equation becomes homogeneous,

$$M\frac{d^2x}{dt^2} + \beta\frac{dx}{dt} + kx = 0. \tag{5.6}$$

We shall see that three types of motion occur called *underdamped*, *critically damped*, and *overdamped*. We illustrate with an example of each before giving a general discussion.

Example 5.3 A 50-gram mass is suspended vertically from a very loose spring with constant 5 newtons per metre. The mass is pulled 5 centimetres below its equilibrium position and given velocity 2 metres per second upward. If, during the motion, the mass is acted on by a damping force in newtons numerically equal to one-tenth the instantaneous velocity in metres per second, find the position of the mass at any time.

Solution If we choose $x = 0$ at the equilibrium position of the mass and x positive upward, the differential equation for the position $x(t)$ of the mass is

$$\frac{50}{1000}\frac{d^2x}{dt^2} + \frac{1}{10}\frac{dx}{dt} + 5x = 0, \qquad \text{or,} \qquad \frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 100x = 0,$$

along with the initial conditions $x(0) = -1/20$, $x'(0) = 2$. The auxiliary equation $m^2 + 2m + 100 = 0$ has solutions

$$m = \frac{-2 \pm \sqrt{4 - 400}}{2} = -1 \pm 3\sqrt{11}i.$$

Consequently,

$$x(t) = e^{-t}[C_1 \cos(3\sqrt{11}t) + C_2 \sin(3\sqrt{11}t)].$$

The initial conditions require

$$-\frac{1}{20} = C_1, \qquad 2 = -C_1 + 3\sqrt{11}C_2,$$

from which $C_2 = 13\sqrt{11}/220$. The position of the mass is therefore given by

$$x(t) = e^{-t}\left[-\frac{1}{20}\cos(3\sqrt{11}t) + \frac{13\sqrt{11}}{220}\sin(3\sqrt{11}t)\right] \text{ m}.$$

The graph of this function in Figure 5.8 clearly indicates how the oscillations decrease in time. This is an example of **underdamped motion.**•

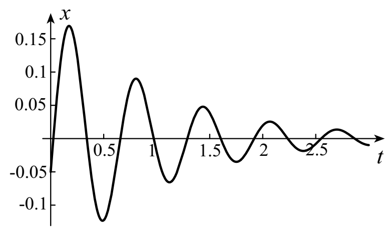


Figure 5.8

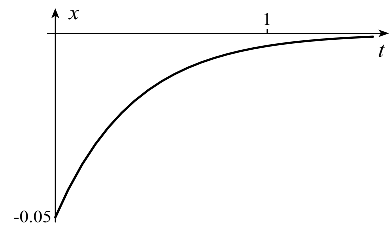


Figure 5.9

Example 5.4 Repeat Example 5.3 if the damping constant is $\beta = 2$.

Solution The differential equation for the position $x(t)$ of the mass is

$$\frac{50}{1000}\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 5x = 0, \qquad \text{or,} \qquad \frac{d^2x}{dt^2} + 40\frac{dx}{dt} + 100x = 0,$$

along with the same initial conditions. The auxiliary equation $m^2 + 40m + 100 = 0$ has solutions

$$m = \frac{-40 \pm \sqrt{1600 - 400}}{2} = -20 \pm 10\sqrt{3}.$$

Consequently,

$$x(t) = C_1 e^{(-20+10\sqrt{3})t} + C_2 e^{(-20-10\sqrt{3})t}.$$

The initial conditions require

$$-\frac{1}{20} = C_1 + C_2, \qquad 2 = (-20 + 10\sqrt{3})C_1 + (-20 - 10\sqrt{3})C_2,$$

from which $C_1 = (2\sqrt{3} - 3)/120$ and $C_2 = -(2\sqrt{3} + 3)/120$. The position of the mass is therefore given by

$$x(t) = \left(\frac{2\sqrt{3} - 3}{120}\right) e^{(-20+10\sqrt{3})t} - \left(\frac{2\sqrt{3} + 3}{120}\right) e^{-(20+10\sqrt{3})t} \text{ m}.$$

The graph of this function is shown in Figure 5.9. This is an example of **overdamped motion**; damping is so large that oscillations are completely eliminated. The mass simply returns to the equilibrium position without passing through it.•

Example 5.5 Repeat Example 5.3 if the damping constant is $\beta = 1$.

Solution The differential equation for the position $x(t)$ of the mass is

$$\frac{50}{1000}\frac{d^2x}{dt^2} + \frac{dx}{dt} + 5x = 0, \qquad \text{or,} \qquad \frac{d^2x}{dt^2} + 20\frac{dx}{dt} + 100x = 0,$$

along with the initial conditions $x(0) = -1/20$, $x'(0) = 2$. The auxiliary equation $m^2 + 20m + 100 = (m + 10)^2 = 0$ has a repeated solution $m = -10$. Consequently,

$$x(t) = (C_1 + C_2 t)e^{-10t}.$$

The initial conditions require

$$-\frac{1}{20} = C_1, \qquad 2 = -10C_1 + C_2,$$

from which $C_2 = 3/2$. The position of the mass is therefore given by

$$x(t) = \left(-\frac{1}{20} + \frac{3t}{2}\right) e^{-10t} \text{ m.}$$

The graph of this function is shown in Figure 5.10. This is an example of **critically damped motion**; any smaller value of the damping constant leads to underdamped motion, and any higher value leads to overdamped motion.•

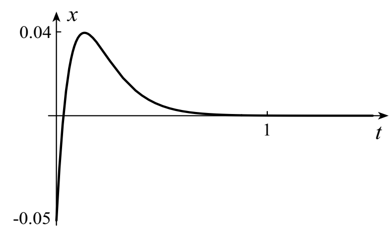


Figure 5.10

We now give a general discussion of differential equation 5.6, clearly delineating values of the parameters M, k, and β that lead to underdamped, critically damped, and overdamped motion. The auxiliary equation associated with

$$M\frac{d^2x}{dt^2} + \beta\frac{dx}{dt} + kx = 0 \tag{5.6}$$

is the quadratic equation

$$Mm^2 + \beta m + k = 0, \tag{5.7a}$$

with solutions

$$m = \frac{-\beta \pm \sqrt{\beta^2 - 4kM}}{2M}. \tag{5.7b}$$

Clearly there are three possibilities depending on the value of $\beta^2 - 4kM$.

Underdamped Motion $\beta^2 - 4kM < 0$

When $\beta^2 - 4kM < 0$, roots 5.7b of the auxiliary equation are complex,

$$m = -\frac{\beta}{2M} \pm \frac{\sqrt{4kM - \beta^2}}{2M}i, \tag{5.8}$$

and a general solution of differential equation 5.6 is

$$x(t) = e^{-\beta t/(2M)} \left[C_1 \cos \frac{\sqrt{4kM - \beta^2}}{2M}t + C_2 \sin \frac{\sqrt{4kM - \beta^2}}{2M}t\right]. \tag{5.9}$$

The presence of the exponential $e^{-\beta t/(2M)}$ before the trigonometric functions indicates that we have damped oscillations. Except possibly for the starting value and initial slope, a typical graph of this function is shown in Figure 5.11.

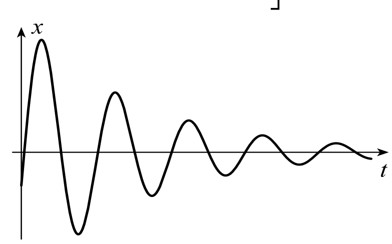


Figure 5.11

Overdamped Motion $\beta^2 - 4kM > 0$

When $\beta^2 - 4kM > 0$, roots 5.7b of the auxiliary equation are real and distinct. A general solution of differential equation 5.6 is

$$x(t) = C_1 e^{(-\beta+\sqrt{\beta^2-4kM})t/(2M)} + C_2 e^{(-\beta-\sqrt{\beta^2-4kM})t/(2M)}. \tag{5.10}$$

In this case the damping factor is so large that oscillations are eliminated and the mass returns from its initial position to the equilibrium position passing through the equilibrium position at most once. Except possibly for starting values and initial slopes, typical graphs of this function are shown in Figure 5.12.

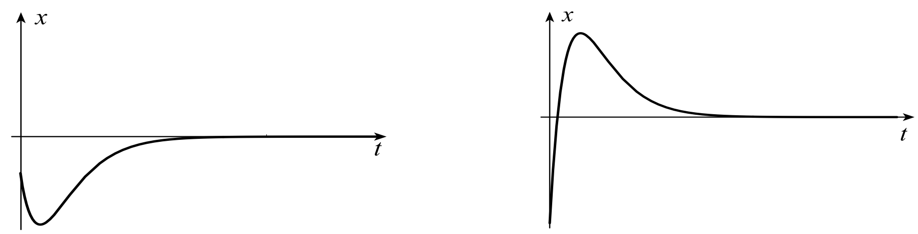


Figure 5.12

Critically Damped Motion $\beta^2 - 4kM = 0$

When $\beta^2 - 4kM = 0$, roots 5.7b of the auxiliary equation are real and equal $m = -\beta/(2M)$, and a general solution of differential equation 5.6 is

$$x(t) = (C_1 + C_2 t)e^{-\beta t/(2M)}. \tag{5.11}$$

Once again no oscillations occur. This situation forms the division between overdamped and underdamped motion. Any increase of β results in overdamped motion and any decrease results in underdamped oscillations. Graphs of this function are similar to those in Figure 5.12.

We now consider further examples of these three possibilities.

Example 5.6 A 100-gram mass is suspended vertically from a spring with constant 5 newtons per metre. The mass is pulled 5 centimetres below its equilibrium position and given velocity 2 metres per second upward. If, during motion, the mass is acted on by a damping force in newtons numerically equal to one-twentieth the instantaneous velocity in metres per second, find the position of the mass at any time.

Solution If we choose $x = 0$ at the equilibrium position of the mass and x positive upward, the differential equation for the position $x(t)$ of the mass is

$$\frac{1}{10}\frac{d^2x}{dt^2} + \frac{1}{20}\frac{dx}{dt} + 5x = 0, \qquad \text{or,} \qquad 2\frac{d^2x}{dt^2} + \frac{dx}{dt} + 100x = 0,$$

along with the initial conditions $x(0) = -1/20$, $x'(0) = 2$. The auxiliary equation $2m^2 + m + 100 = 0$ has solutions

$$m = \frac{-1 \pm \sqrt{1-800}}{4} = \frac{-1 \pm \sqrt{799}i}{4}.$$

Consequently,

$$x(t) = e^{-t/4}[C_1 \cos(\sqrt{799}t/4) + C_2 \sin(\sqrt{799}t/4)].$$

The initial conditions require

$$-\frac{1}{20} = C_1, \qquad 2 = -\frac{C_1}{4} + \frac{\sqrt{799}C_2}{4},$$

from which $C_2 = 159\sqrt{799}/15980$. The position of the mass is therefore given by

$$x(t) = e^{-t/4}\left[-\frac{1}{20}\cos\left(\frac{\sqrt{799}t}{4}\right) + \frac{159\sqrt{799}}{15980}\sin\left(\frac{\sqrt{799}t}{4}\right)\right] \text{ m.}$$

Using the technique suggested in Example 5.2, we can rewrite the displacement in the form

$$x(t) = Ae^{-t/4}\sin\left(\frac{\sqrt{799}t}{4} + \phi\right),$$

where

$$A = \sqrt{\left(-\frac{1}{20}\right)^2 + \left(\frac{159\sqrt{799}}{15980}\right)^2} =\approx 0.285661.$$

The graph of these underdamped oscillations is shown in Figure 5.13. Oscillations are bounded by the curves $x = \pm 0.285661e^{-t/4}$, shown dotted.•

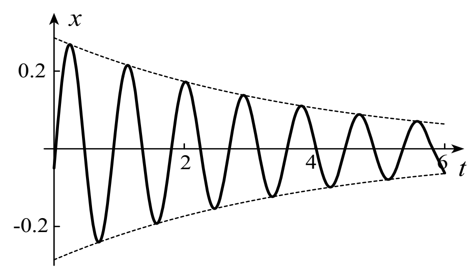


Figure 5.13

Example 5.7 A 4-kilogram mass is attached to a horizontal spring. The mass moves on a frictionless surface, but a dashpot creates a damping force in newtons equal to ten times the velocity of the mass. What spring constant leads to critically damped motion?

Solution Critically damped motion results when spring constant k, mass $M = 4$, and damping factor $\beta = 10$ are related by $\beta^2 - 4kM = 0$; that is, $100 - 4k(4) = 0$. This implies that $k = 25/4$ N/m.•

Example 5.8 A 2-kilogram mass is suspended vertically from a spring with constant 500 newtons per metre. The mass is pulled 10 centimetres below its equilibrium position and given velocity 5 metres per second downward. A dashpot is attached to the mass creating a damping force in newtons numerically equal to one hundred times the instantaneous velocity in metres per second. Show that motion of the mass is overdamped and that in 1 second the mass is within 1 millimetre of its equilibrium position.

Solution If we choose $x = 0$ at the equilibrium position of the mass and x positive upward, the initial-value problem for the position $x(t)$ of the mass is

$$2\frac{d^2x}{dt^2} + 100\frac{dx}{dt} + 500x = 0, \qquad x(0) = -\frac{1}{10}, \quad x'(0) = -5.$$

The auxiliary equation $2m^2 + 100m + 500 = 2(m^2 + 50m + 250) = 0$ has solutions

$$m = \frac{-50 \pm \sqrt{2500 - 1000}}{2} = -25 \pm 5\sqrt{15}.$$

With real roots, motion is overdamped and the position function is of the form

$$x(t) = C_1 e^{(-25+5\sqrt{15})t} + C_2 e^{(-25-5\sqrt{15})t}.$$

The initial conditions require

$$-\frac{1}{10} = C_1 + C_2, \qquad -5 = (-25 + 5\sqrt{15})C_1 - (25 + 5\sqrt{15})C_2.$$

These can be solved for

$$C_1 = -\frac{\sqrt{15}+1}{20}, \qquad C_2 = \frac{\sqrt{15}-1}{20}.$$

The position of the mass is therefore given by

$$x(t) = -\left(\frac{\sqrt{15}+1}{20}\right) e^{(-25+5\sqrt{15})t} + \left(\frac{\sqrt{15}-1}{20}\right) e^{(-25-5\sqrt{15})t} \text{ m}.$$

If we set $t = 1$, we obtain the position of the mass after one second,

$$x(1) = -\left(\frac{\sqrt{15}+1}{20}\right) e^{(-25+5\sqrt{15})} + \left(\frac{\sqrt{15}+1}{20}\right) e^{(-25-5\sqrt{15})} = -0.000870 \text{ m};$$

that is, the mass is 0.87 millimetres from the equilibrium position.●

EXERCISES 5.2

1. A 1-kilogram mass is suspended vertically from a spring with constant 16 newtons per metre. The mass is pulled 10 centimetres below its equilibrium position and then released. Find the position of the mass, relative to its equilibrium position, if a damping force in newtons equal to one-tenth the instantaneous velocity in metres per second acts on the mass.

2. Repeat Exercise 1 if the damping force is equal to ten times the instantaneous velocity.

3. What damping factor creates critically damped motion for the spring and mass in Exercise 1?

4. A 100-gram mass is suspended vertically from a spring with constant 4000 newtons per metre. The mass is pulled 2 centimetres above its equilibrium position and given a downward velocity of 4 metres per second. Find the position of the mass, relative to its equilibrium position, if a dashpot is attached to the mass so as to create a damping force in newtons equal to forty times the instantaneous velocity in metres per second. Does the mass ever pass through the equilibrium position?

5. Repeat Exercise 4 if the mass is given a downward velocity of 10 metres per second.

6. (a) A 1-kilogram mass is suspended vertically from a spring with constant 50 newtons per metre. The mass is pulled 5 centimetres above its equilibrium position and given an upward velocity of 3 metres per second. Find the position of the mass, relative to its equilibrium position, if a dashpot is attached to the mass so as to create a damping force in newtons equal to fifteen times the instantaneous velocity in metres per second.
(b) Does the mass ever pass through the equilibrium position?
(c) When is the mass 1 centimetre from the equilibrium position?
(d) Sketch a graph of the position function.

7. Repeat Exercise 6 if the initial velocity is 1 metre per second downward.

8. Repeat Exercise 6 if the initial velocity is 3 metres per second downward.

9. (a) A 2-kilogram mass is suspended vertically from a spring with constant 200 newtons per metre. The mass is pulled 10 centimetres above its equilibrium position and given an upward velocity of 5 metres per second. Find the position of the mass, relative to its equilibrium position, if a damping force in newtons equal to four times the instantaneous velocity in metres per second also acts on the mass.

(b) What is the maximum distance the mass attains from equilibrium?

(c) When does the mass first pass through the equilibrium position?

10. (a) A 1-kilogram mass is suspended vertically from a spring with constant 40 newtons per metre. The mass is pulled 5 centimetres below its equilibrium position and released. Find the position of the mass, relative to its equilibrium position, if a dashpot is attached to the mass so as to create a damping force in newtons equal to twice the instantaneous velocity in metres per second. Express the function in the form $Ae^{-at}\sin(\omega t + \phi)$ for appropriate a, A, ω, and ϕ.

(b) Show that the length of time between successive passes through the equilibrium position is constant. What is this time? Twice its value is often called the **quasi period** for overdamped motion? Is it the same as the period of the corresponding undamped system?

11. A mass M is suspended from a spring with constant k. Motion is initiated by giving the mass a displacement x_0 from equilibrium and a velocity v_0. A damping force with constant $\beta > 0$ results in critically damped motion.

(a) Show that if x_0 and v_0 are both positive or both negative, the mass cannot pass through its equilibrium position.

(b) When x_0 and v_0 have opposite signs, it is possible for the mass to pass through the equilibrium position, but it can do so only once. What condition must x_0 and v_0 satisfy for this to happen?

12. A mass M is suspended from a spring with constant k. Motion is initiated by giving the mass a displacement x_0 from equilibrium and a velocity v_0. A damping force with constant $\beta > 0$ results in overdamped motion.

(a) Show that if x_0 and v_0 are both positive or both negative, the mass cannot pass through its equilibrium position.

(b) When x_0 and v_0 have opposite signs, it is possible for the mass to pass through the equilibrium position, but it can do so only once. What condition must x_0 and v_0 satisfy for this to happen?

13. A weighing platform has weight W and is supported by springs with combined spring constant k. A package with weight w is dropped on the platform so that the two move together. Find a formula for the maximum value of w so that oscillations do not occur. Assume that there is damping in the motion with constant β.

14. A mass M is suspended from a spring with constant k. Oscillations are initiated by giving the mass a displacement x_0 from equilibrium and a velocity v_0. A damping force with constant $\beta > 0$ results in underdamped motion.

(a) Show that the position of the mass relative to its equilibrium position can be expressed in the form

$$x(t) = Ae^{-\beta t/(2M)}\sin\left(\frac{\sqrt{4kM - \beta^2}}{2M}t + \phi\right),$$

where A and ϕ are constants.

(b) Show that the length of time between successive passes through the equilibrium position is constant. What is this time?

(c) Let t_1, t_2, ..., be the times at which the velocity of the mass is equal to zero (and therefore the times at which $x(t)$ has relative maxima and minima. If x_1, x_2, ..., are the corresponding values of $x(t)$, show that the ratio

$$\frac{x_n}{x_{n+2}} = e^{2\pi\beta/\sqrt{4kM-\beta^2}},$$

is a constant independent of n. The quantity $2\pi\beta/\sqrt{4kM-\beta^2}$ is called the **logarithmic decrement**.

5.3 Vibrating Mass-Spring Systems With External Forces

So far in this chapter we have considered mass-spring systems with damping forces, and in the case of vertical oscillations, gravity is also a consideration. With only these forces, the differential equation describing motion is homogeneous. Problems become more interesting, and more widely applicable, when other forces are taken into consideration. In particular, periodic forcing functions lead to *resonance*.

When all other forces acting on the mass in a damped mass-spring system are grouped together into one term denoted by $F(t)$, the differential equation describing motion is

$$M\frac{d^2x}{dt^2} + \beta\frac{dx}{dt} + kx = F(t). \tag{5.12}$$

We consider various possibilities for $F(t)$. To begin with, you may have noticed that in every example of masses sliding along horizontal surfaces (Figure 5.14), we have ignored friction between the mass and the surface. Suppose we now take it into account. If the coefficient of kinetic friction between the mass and surface is μ (see Section 3.2), then the force of friction retarding motion has magnitude μMg where $g > 0$ is the acceleration due to gravity. Entering this force into differential equation 5.12 for all time is a problem due to the difficulty in specifying the direction of the force. Certainly we can say that friction is always in a direction opposite to velocity, but how do we represent this mathematically without destroying the linearity of differential equation 5.12. We can't for all time. It is necessary to reconstitute the differential equation each time the mass changes direction. The following example is an illustration.

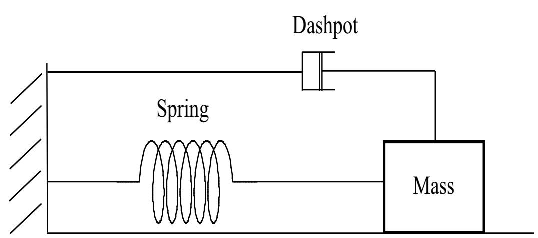


Figure 5.14

Example 5.9 A 1-kilogram mass, attached to a spring with constant 16 newtons per metre, slides horizontally along a surface where the coefficient of kinetic friction between surface and mass is $\mu = 1/10$. Motion is initiated by pulling the mass 10 centimetres to the right of its equilibrium position and giving it velocity 1 metre per second to the left. If any damping forces are negligible, find the point where the mass comes to an instantaneous stop for the second time.

Solution While the mass is travelling to the left for the first time, the force of friction is to the right, and therefore the initial-value problem for its position during this time is

$$\frac{d^2x}{dt^2} + 16x = \left(\frac{1}{10}\right)(1)g, \qquad x(0) = \frac{1}{10}, \quad x'(0) = -1,$$

where $g = 9.81$. Since the auxiliary equation $m^2 + 16 = 0$ has roots $m = \pm 4i$, a general solution of the associated homogeneous differential equation is $x_h(t) =$

$C_1 \cos 4t + C_2 \sin 4t$. It is easy to spot that a particular solution of the nonhomogeneous equation is $x_p(t) = g/160$, and therefore a general solution of the nonhomogeneous differential equation is

$$x(t) = C_1 \cos 4t + C_2 \sin 4t + \frac{g}{160}.$$

The initial conditions require

$$\frac{1}{10} = C_1 + \frac{g}{160}, \qquad -1 = 4C_2.$$

Hence,

$$x(t) = \left(\frac{1}{10} - \frac{g}{160}\right)\cos 4t - \frac{1}{4}\sin 4t + \frac{g}{160} \text{ m}.$$

This represents the position of the mass only while it is travelling to the left for the first time. To determine the time and place at which the mass stops moving to the left, we set the velocity equal to zero,

$$0 = \frac{dx}{dt} = -4\left(\frac{1}{10} - \frac{g}{160}\right)\sin 4t - \cos 4t.$$

This equation can be simplified to

$$\tan 4t = \frac{40}{g - 16},$$

solutions of which are

$$t = \frac{1}{4}\,\mathrm{Tan}^{-1}\!\left(\frac{40}{g-16}\right) + \frac{n\pi}{4},$$

where n is an integer. The only acceptable solution is the smallest positive one, and this occurs for $n = 1$, giving $t = 0.431082$ s. The position of the mass at this time is $x(0.431082) = -0.191663$ m. The mass will move from this position if the spring force is sufficient to overcome the force of static friction. Let us suppose that the coefficient of static friction is $\mu_s = 1/5$ (see Section 3.2). This means that the smallest force necessary for the mass to move has magnitude $(1/5)(1)(9.81) = 1.962$ N. Since the spring force at the first stopping position is $0.191663(16) = 3.06661$ N, it is more than enough to overcome the force of static friction.

For the return trip to the right, friction is to the left, and therefore the initial-value problem for position is

$$\frac{d^2x}{dt^2} + 16x = -\frac{g}{10}, \qquad x(0) = -0.191663, \quad x'(0) = 0.$$

For simplicity, we have reinitialized time $t = 0$ to commencement of motion to the right (see Exercise 1 for the analysis without reinitializing time). A general solution of this differential equation is

$$x(t) = C_3 \cos 4t + C_4 \sin 4t - \frac{g}{160}.$$

The initial conditions require

$$-0.191663 = C_3 - \frac{g}{160}, \qquad 0 = 4C_4.$$

Thus,

$$x(t) = \left(\frac{g}{160} - 0.191663\right)\cos 4t - \frac{g}{160}\text{ m.}$$

The mass comes to rest when

$$0 = \frac{dx}{dt} = -4\left(\frac{g}{160} - 0.191663\right)\sin 4t,$$

solutions of which are given by $t = n\pi/4$ where n is an integer. The smallest positive value is $t = \pi/4$ and the position of the mass at this time is

$$x(\pi/4) = \left(\frac{g}{160} - 0.191663\right)\cos \pi - \frac{g}{160} = 0.069038\text{ m;}$$

that is, the mass is 6.9 cm to the right of the equilibrium position. The spring force is still sufficient to overcome the force of friction and the mass will again move to the left.•

Periodic Forcing Functions and Resonance

We now consider the application of periodic forcing functions to masses on the ends of springs. When an external force $F(t) = A\sin\omega t$, where $A > 0$ and $\omega > 0$ are constants, acts on the mass in a mass-spring system, differential equation 5.12 describing motion becomes

$$M\frac{d^2x}{dt^2} + \beta\frac{dx}{dt} + kx = A\sin\omega t. \tag{5.13}$$

We begin discussions with systems that have no damping, somewhat unrealistic perhaps, but essential ideas are not obscured by intensive calculations. The next example introduces the general discussion to follow.

Example 5.10 A 2-kilogram mass is suspended from a spring with constant 128 newtons per metre. It is pulled 4 centimetres above its equilibrium position and released. An external force $3\sin\omega t$ newtons acts on the mass during its motion. If damping is negligible, find the position of the mass as a function of time.

Solution The initial-value problem for position of the mass is

$$2\frac{d^2x}{dt^2} + 128x = 3\sin\omega t, \qquad x(0) = 1/25, \quad x'(0) = 0.$$

Because the auxiliary equation $2m^2 + 128 = 0$ has solutions $m = \pm 8i$, a general solution of the associated homogeneous differential equation is $x_h(t) = C_1\cos 8t + C_2\sin 8t$. Undetermined coefficients suggests a particular solution of the form $x_p(t) = A\sin\omega t + B\cos\omega t$. Substitution into the differential equation leads to $x_p(t) = [3/(128 - 2\omega^2)]\sin\omega t$. Thus, a general solution of the nonhomogeneous differential equation is

$$x(t) = C_1\cos 8t + C_2\sin 8t + \frac{3}{128 - 2\omega^2}\sin\omega t.$$

The initial conditions require

$$\frac{1}{25} = C_1, \qquad 0 = 8C_2 + \frac{3\omega}{128 - 2\omega^2}.$$

Thus, the position of the mass at any time is

$$x(t) = \frac{1}{25}\cos 8t + \frac{3\omega}{16(\omega^2 - 64)}\sin 8t + \frac{3}{2(64 - \omega^2)}\sin \omega t \text{ m.}$$

But if $\omega = 8$ the last two terms have vanishing denominators. We should have made allowances for this when determining the particular solution. When $\omega = 8$, the right side of the differential equation is a part of $x_h(t)$, and therefore we should take $x_p(t) = t(A\sin 8t + B\cos 8t)$. Substitution into the differential equation leads to $x_p(t) = -(3t/32)\cos 8t$, and therefore a general solution of the differential equation with nonhomogeneity $3\sin 8t$ is

$$x(t) = C_1\cos 8t + C_2\sin 8t - \frac{3t}{32}\cos 8t.$$

The initial conditions require

$$\frac{1}{25} = C_1, \qquad 0 = 8C_2 - \frac{3}{32}.$$

The position of the mass when the forcing function is $3\sin 8t$ is

$$x(t) = \frac{1}{25}\cos 8t + \frac{3}{256}\sin 8t - \frac{3t}{32}\cos 8t \text{ m.}$$

A graph of this function is shown in Figure 5.15. The last term in the solution has led to oscillations that become unbounded. This is a direct result of the fact that when $\omega = 8$, the frequency of the forcing term is equal to the frequency at which the system would oscillate were no forcing term present (the so-called natural frequency of the system). (Think of this as similar to a parent pushing a child on a swing. Every other time the swing begins its downward motion, the parent applies a force, resulting in the child going higher and higher. The parent applies the force at the same frequency as the motion of the swing.)•

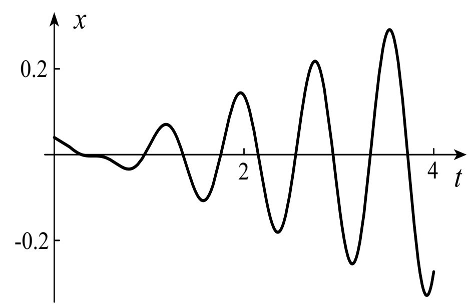


Figure 5.15

This phenomenon of ever increasing oscillations due to a forcing function with the same frequency as the natural frequency of the system is known as **resonance**. Because the system is undamped, we refer to this as **undamped resonance**.

Let us discuss resonance for the general undamped mass-spring system. When a periodic force $A\sin\omega t$ is applied to an undamped mass-spring system, the differential equation describing motion is

$$M\frac{d^2x}{dt^2} + kx = A\sin\omega t. \tag{5.14}$$

When $\omega \neq \sqrt{k/M}$, the natural frequency of the system and the forcing frequency are different. A general solution of the differential equation takes the form

$$x(t) = C_1\cos\left(\sqrt{k/M}t\right) + C_2\sin\left(\sqrt{k/M}t\right) + \frac{A}{k - M\omega^2}\sin\omega t, \tag{5.15}$$

and there is nothing untoward about oscillations. When $\omega = \sqrt{k/M}$, so that the forcing frequency is identical to the natural frequency of the undamped system, differential equation 5.14 takes the form

$$M\frac{d^2x}{dt^2} + kx = A\sin\left(\sqrt{k/M}t\right), \tag{5.16}$$

In this case the general solution

$$x(t) = C_1\cos\left(\sqrt{k/M}t\right) + C_2\sin\left(\sqrt{k/M}t\right) - \frac{At}{2\sqrt{kM}}\cos\left(\sqrt{k/M}t\right) \tag{5.17}$$

exhibits undamped resonance.

Resonance also occurs in damped systems, but there is a difference; oscillations can become large depending on the degree of damping, but they cannot become unbounded. Differential equation 5.13 describes motion of a damped mass-spring system in the presence of a periodic forcing function. Equations 5.9–5.11 define general solutions of the associated homogeneous equation, and it is clear that none of these solutions contain the nonhomogeneity $A\sin\omega t$ for any ω. To put it another way, in the presence of damping, simple harmonic motion is not possible, and therefore the system does not have a natural frequency. Resonance as found in undamped systems is therefore not possible. For underdamped motion, however, oscillations can be large, depending on the degree of damping, and this is again known as resonance, but we call it **damped resonance**. We illustrate in the following example.

Example 5.11 A 1-kilogram mass is at rest, suspended from a spring with constant 65 newtons per metre. Attached to the mass is a dashpot that creates a damping force equal to twice the velocity of the mass whenever the mass is in motion. At time $t = 0$, a vertical force $3\sin\omega t$ begins to act on the mass. Find the position function for the mass. For what value of ω are oscillations largest?

Solution The initial-value problem for the motion of the mass is

$$\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 65x = 3\sin\omega t, \qquad x(0) = 0, \quad x'(0) = 0.$$

The auxiliary equation $m^2 + 2m + 65 = 0$ has solutions $m = -1 \pm 8i$ so that a general solution of the associated homogeneous differential equation is $x_h(t) = e^{-t}(C_1\cos 8t + C_2\sin 8t)$. A particular solution can be found in the form $x_p(t) = A\sin\omega t + B\cos\omega t$ by undetermined coefficients. The result is

$$x_p(t) = \frac{3(65-\omega^2)}{(65-\omega^2)^2 + 4\omega^2}\sin\omega t - \frac{6\omega}{(65-\omega^2)^2 + 4\omega^2}\cos\omega t.$$

A general solution of the nonhomogeneous differential equation is therefore

$$x(t) = e^{-t}(C_1\cos 8t + C_2\sin 8t) + \frac{3}{(65-\omega^2)^2 + 4\omega^2}\left[(65-\omega^2)\sin\omega t - 2\omega\cos\omega t\right].$$

The initial conditions require

$$0 = C_1 - \frac{6\omega}{(65-\omega^2)^2 + 4\omega^2}, \qquad 0 = -C_1 + 8C_2 + \frac{3\omega(65-\omega^2)}{(65-\omega^2)^2 + 4\omega^2}.$$

These give

$$C_1 = \frac{6\omega}{(65-\omega^2)^2+4\omega^2}, \qquad C_2 = \frac{3\omega(\omega^2-63)}{8[(65-\omega^2)^2+4\omega^2]},$$

and the position of the mass is therefore

$$x(t) = \frac{3\omega e^{-t}}{8[(65-\omega^2)^2+4\omega^2]}[16\cos 8t + (\omega^2-63)\sin 8t] + \frac{3}{(65-\omega^2)^2+4\omega^2}\left[(65-\omega^2)\sin\omega t - 2\omega\cos\omega t\right] \text{ m.}$$

The terms involving $\cos 8t$ and $\sin 8t$ are called the **transient** part of the solution, transient because the e^{-t} factor effectively eliminates these terms after a long time. The terms involving $\sin\omega t$ and $\cos\omega t$, not being subjected to such a factor, do not diminish in time. They are called the **steady-state** part of the solution. In Figure 5.16a we have shown the transient solution; Figure 5.16b shows the steady-state solution with the specific choice $\omega = 4$; and Figure 5.16c shows their sum.

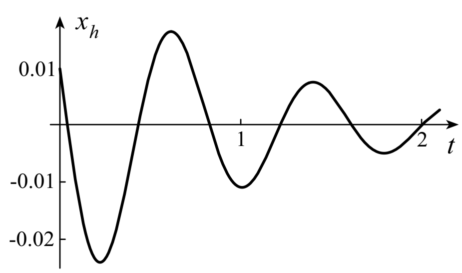


Figure 5.16a

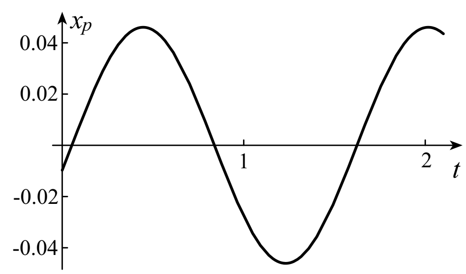


Figure 5.16b

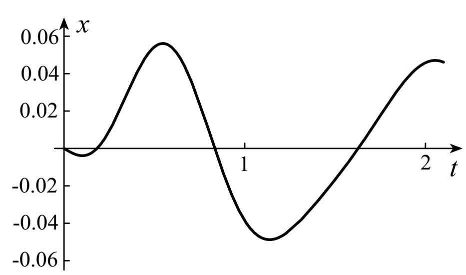


Figure 5.16c

When the forcing frequency is equal to the natural frequency in undamped systems, resonance in the form of unbounded oscillations occurs. Inspection of the above solution indicates that for no value of ω can oscillations become unbounded in this damped system. However, there is a value of ω that makes oscillations largest relative to all other values of ω. In particular, because the transient part of the solution becomes negligible after a sufficiently long time, we are interested in maximizing the amplitude of the steady-state part of the solution. It is the particular solution $x_p(t)$. The amplitude of the oscillations represented by this term is

$$\sqrt{\left[\frac{3(65-\omega^2)}{(65-\omega^2)^2+4\omega^2}\right]^2 + \left[\frac{-6\omega}{(65-\omega^2)^2+4\omega^2}\right]^2} = \frac{3}{\sqrt{(65-\omega^2)^2+4\omega^2}};$$

that is, the steady-state solution can be expressed in the form

$$x_p(t) = \frac{3}{\sqrt{(65-\omega^2)^2+4\omega^2}} \sin(\omega t + \phi)$$

for some ϕ. To maximize the amplitude we minimize $(65-\omega^2)^2+4\omega^2$. Setting its derivative equal to zero gives

$$0 = 2(65-\omega^2)(-2\omega) + 8\omega,$$

and the only positive solution of this equation is $\omega = 3\sqrt{7}$. For this value of ω, the steady-state solution becomes

$$x_p(t) = \frac{3}{16}\sin(3\sqrt{7}t).$$

Maximum oscillations have been realized and the system is said to be in damped resonance. We have shown a graph of this function in Figure 5.17. Compare the scale on the vertical axis in this figure to that in Figure 5.16b where $\omega = 4$. We have shown a plot of amplitude versus ω in Figure 5.18.•

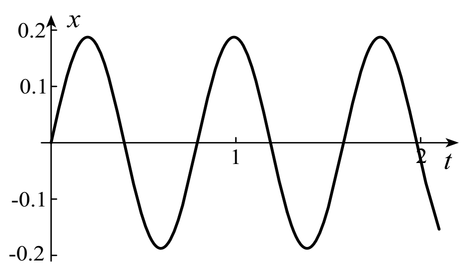


Figure 5.17

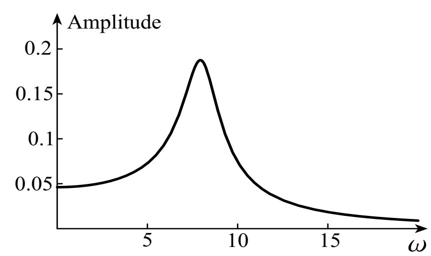


Figure 5.18

For a general discussion of resonance in damped systems, see Exercise 13.

Amplitude Modulation

Suppose an oscillatory system is modelled by differential equation 5.2 with no damping and an external forcing function $A\cos\omega t$, where A and ω are positive constants. Provided ω is not equal to the natural frequency $\omega_0 = \sqrt{k/M}$ of the system, it is straightforward to show that a general solution of this equation is

$$x(t) = C_1\cos\omega_0 t + C_2\sin\omega_0 t + \frac{A}{k-M\omega^2}\cos\omega t.$$

If the system has no initial energy at time $t=0$, and the force $A\cos\omega t$ excites the system, then the initial conditions are $x(0)=0$ and $x'(0)=0$, and these imply that $C_2 = 0$ and $C_1 = A/(k-M\omega^2)$. Thus,

$$x(t) = \frac{A}{M(\omega_0^2-\omega^2)}(\cos\omega t - \cos\omega_0 t).$$

By using the trigonometric identity $\cos F - \cos G = -2\sin\left(\frac{F+G}{2}\right)\sin\left(\frac{F-G}{2}\right)$, we can write this solution in the form

$$x(t) = \frac{2A}{M(\omega^2-\omega_0^2)}\sin\left(\frac{\omega-\omega_0}{2}\right)t\sin\left(\frac{\omega+\omega_0}{2}\right)t.$$

If ω is very close to ω_0 (they have been assumed not equal), then $|\omega_0 - \omega|$ is very much smaller than $\omega_0 + \omega$. Suppose for instance that $\omega = 1.1\omega_0$, in which case

$$\begin{aligned} x(t) &= \frac{2A}{M(1.1^2\omega_0^2 - \omega_0^2)} \sin\left(\frac{1.1\omega_0 - \omega_0}{2}\right) t \sin\left(\frac{1.1\omega_0 + \omega_0}{2}\right) t \\ &= \frac{9.52A}{M\omega_0} \sin(0.05\omega_0 t) \sin(1.05\omega_0 t). \end{aligned}$$

We have plotted this function in Figure 5.19a. We have also included plots of the functions $\pm\dfrac{9.52A}{M\omega_0}\sin(0.05\omega_0 t)$ (the dashed curves), and have done so for the following reason. Obviously the period of the dashed curves is much larger than that of $x(t)$. We could look at the function $x(t)$ as the sine function $\sin(1.05\omega_0 t)$ with a time varying amplitude $\pm\dfrac{9.52A}{M\omega_0}\sin(0.05\omega_0 t)$. In Figure 5.19b, we have plotted the same curves when $\omega = 1.05\omega_0$, in which case

$$x(t) = \frac{19.5}{M\omega_0^2} \sin(0.025\omega_0 t) \sin(1.025\omega_0 t).$$

The larger frequency $1.025\omega_0$ has not changed much from $1.05\omega_0$, but the smaller frequency has doubled. In addition, there is a substantial increase in the time varying amplitude; it has more than doubled. What we are seeing in these graphs are called **beats**. Beats can actually be heard when two musical instruments produce sounds with frequencies that are very close to each other. In electronics, this is called **amplitude modulation**.

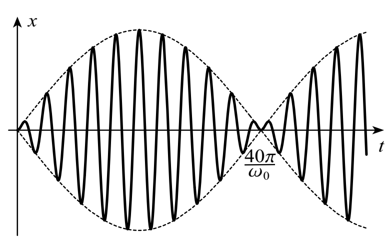


Figure 5.19a

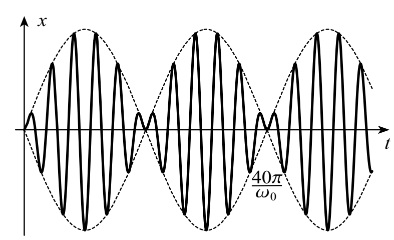


Figure 5.19b

EXERCISES 5.3

1. Repeat Example 5.9 without reinitializing time for movement to the right.

2. A 0.5-kilogram mass sits on a table attached to a spring with constant 18 newtons per metre (Figure 5.14). The mass is pulled so as to stretch the spring 6 centimetres and then released.

(a) If friction between the mass and the table creates a force of 0.5 newtons that opposes motion, but damping is negligible, show that the differential equation determining motion is

$$\frac{d^2x}{dt^2} + 36x = 1, \quad x(0) = 0.06, \; x'(0) = 0.$$

Assume that the coefficient of static friction is twice the coefficient of kinetic friction.

(b) Find where the mass comes to rest for the first time. Will it move from this position?

3. Repeat Exercise 2 given that the mass is pulled 25 centimetres to the right.

4. A 200-gram mass rests on a table attached to an unstretched spring with constant 5 newtons per metre. The mass is given a velocity of 1/2 metre per second to the right. During the subsequent motion, the coefficient of kinetic friction between mass and table is $\mu_k = 1/4$, but damping is negligible. Where does the mass come to a complete stop? Assume that the coefficient of static friction is $\mu_s = 1/2$.

5. Repeat Exercise 4 if the initial velocity is 2 metres per second.

6. A 100-gram mass is suspended from a spring with constant 4000 newtons per metre. At its equilibrium position, it is suddenly (time $t = 0$) given an upward velocity of 10 metres per second. If an external force $3\cos 100t$, $t \geq 0$ acts on the mass, find its displacement as a function of time. Does resonance occur?

7. Repeat Exercise 6 if the external force is $3\cos 200t$.

8. A vertical spring having constant 64 newtons per metre has a 1-kilogram mass attached to it. An external force $2\sin 4t$, $t \geq 0$ is applied to the mass. If the mass is at rest at its equilibrium position at time $t = 0$, and damping is negligible, find the position of the mass as a function of time. Does resonance occur?

9. Repeat Exercise 8 if the external force is $2\sin 8t$.

10. A mass M is suspended from a vertical spring with constant k. If an external force $F(t) = A\cos\omega t$ is applied to the mass for $t > 0$, find the value of ω that causes resonance.

11. A 200-gram mass suspended vertically from a spring with constant 10 newtons per metre is set into vibration by an external force in newtons given by $4\sin 10t$, $t \geq 0$. During the motion a damping force in newtons equal to 3/2 the velocity on the mass in metres per second acts on the mass. Find the position of the mass as a function of time t.

12. (a) A 1-kilogram mass is motionless, suspended from a spring with constant 100 newtons per metre. A vertical force $2\sin\omega t$ acts on the mass beginning at time $t = 0$. Oscillations are subject to a damping force in newtons equal to twice the velocity in metres per second. Find the position of the mass as a function of time.

(b) What value of ω causes resonance? What is the amplitude of steady-state oscillations for resonance?

13. A mass M is suspended from a spring with constant k. Vertical motion is initiated by an external force $A\cos\omega t$ where A is a positive constant. During the subsequent motion a damping force acts on the mass with damping coefficient β.

(a) Show that the steady-state part of the solution is

$$x_p(t) = \frac{A(k - M\omega^2)}{(k - M\omega^2)^2 + \beta^2\omega^2}\cos\omega t - \frac{A\omega\beta}{(k - M\omega^2)^2 + \beta^2\omega^2}\sin\omega t.$$

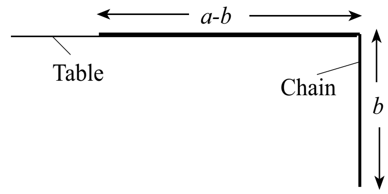
(b) Find the value of ω that gives resonance and the resulting amplitude of oscillations.

14. A battery of springs is placed between two sheets of wood, and the structure is placed on a level floor. Equivalent to the springs is a single spring with constant 1000 newtons per metre. A 20 kilogram mass is dropped onto the upper platform, hitting the platform with speed 2 metres per second, and remains attached to the platform thereafter.

(a) Find the position of the mass relative to where it strikes the platform as a function of time. Assume that air drag is 10 times the velocity of the mass.

(b) What is the maximum displacement from where it strikes the platform experienced by the mass?

15. A uniform chain of length a has portion $0 < b < a$ hanging over the edge of a smooth table (figure to the right). Find the time taken for the chain to slide off the table if it starts from rest.



16. (a) If the coefficient of static friction between the chain and table in Exercise 15 is μ_s, what is the smallest value of b for motion to commence?
(b) Assuming that the condition in part (a) is met, and that the coefficient of kinetic friction between the chain and the table is μ_k, find the time taken for the chain to slide off the table if it starts from rest.

17. A mass of 500 grams is at equilibrium suspended from a spring with constant 250 newtons per metre. At time $t = 0$, the apparatus to which the top end of the spring is attached moves up and down sinusoidally according to $f(t) = 0.1 \sin 2t$ metres, where $f(t)$ is positive when the apparatus is above its starting position. If damping with coefficient $\beta = 10$ acts on the mass during its motion, find the position of the mass as a function of time. Describe the motion of the mass.

18. A mass M, attached to a spring with constant k, rests on a horizontal table. At time $t = 0$ it is pulled to the right a distance x_0 and given velocity v_0 to the right. If damping is ignored, but the coefficient of kinetic friction between table and mass is μ, find a formula for the time when the mass comes to an instantaneous stop for the first time.

19. Repeat Exercise 18 if the initial velocity is to the left.

20. A cube 1 metre on each side and with density 1200 kilograms per cubic metre is placed with one of its faces in the surface of a body of water. When the cube is released from this position and sinks, it is acted upon by three forces, gravity, a buoyant force equal to the weight of water displaced by the submerged portion of the cube (Archimedes' principle), and a resistive force equal to twice the speed of the object. Find the depth of the bottom surface of the cube as a function of time from the instant the cube is released until it is completely submerged. Plot a graph of the function.

21. A cube 1 metre on each side and with density 500 kilograms per cubic metre is placed with one of its faces in the surface of a body of water. When the cube is released from this position and sinks, it is acted upon by three forces, gravity, a buoyant force equal to the weight of water displaced by the submerged portion of the cube (Archimedes' principle), and a resistive force equal to twice the speed of the object. Find the depth of the bottom surface of the cube as a function of time. Plot a graph of the function.

22. A cable hangs over a peg, 10 metres on one side and 15 metres on the other. Find the time for it to slide off the peg
(a) if friction at the peg is negligible.
(b) if friction at the peg is equal to the weight of 1 metre of cable.