Solutions to midterm 2 Fall 2023

1. Evaluate the following limit or explain why it does not exist.

$$\lim_{(x,y)\to(1,1)} \frac{x^3 - xy}{x^3 + xy - 2}$$

If we approach $(1,1)$ along the line $x = 1$,

$$\lim_{(x,y)\to(1,1)} \frac{x^3 - xy}{x^3 + xy - 2} = \lim_{y\to 1} \frac{1-y}{1+y-2} = \lim_{y\to 1} \frac{1-y}{y-1} = -1.$$

If we approach along the line $y = 1$,

$$\lim_{(x,y)\to(1,1)} \frac{x^3 - xy}{x^3 + xy - 2} = \lim_{x\to 1} \frac{x^3 - x}{x^3 + x - 2} = \lim_{x\to 1} \frac{x(x-1)(x+1)}{(x-1)(x^2+x+2)} = \lim_{x\to 1} \frac{x(x+1)}{x^2+x+2} = \frac{2}{4} = \frac{1}{2}.$$

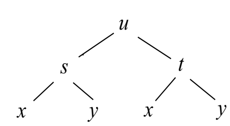
Since these are different, the original limit does not exist.

2. Let $u(x,y) = f(x^3+y)+(x^3+y)(x^3-y)$ where f is a twice differentiable function and let $s = x^3+y$ and $t = x^3 - y$. Prove that

$$\frac{\partial u}{\partial x} - 3x^2\frac{\partial u}{\partial y} + \frac{\partial^2 u}{\partial y \partial x} = 3x^2\big(f''(s) + 2s\big).$$

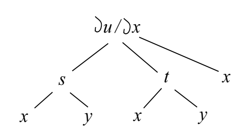
Since $u(x,y) = f(s) + st$, the schematic to the right gives

$$\begin{aligned}\frac{\partial u}{\partial x} &= \frac{\partial u}{\partial s}\frac{\partial s}{\partial x} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial x}\\ &= [f'(s)+t](3x^2) + s(3x^2) = 3x^2[f'(s)+t+s],\\ \frac{\partial u}{\partial y} &= \frac{\partial u}{\partial s}\frac{\partial s}{\partial y} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial y}\\ &= [f'(s)+t](1) + s(-1) = f'(s)+t-s.\end{aligned}$$



The schematic to the right gives

$$\begin{aligned}\frac{\partial^2 u}{\partial y \partial x} &= \frac{\partial}{\partial s}\left(\frac{\partial u}{\partial x}\right)\frac{\partial s}{\partial y} + \frac{\partial}{\partial t}\left(\frac{\partial u}{\partial x}\right)\frac{\partial t}{\partial y}\\ &= [3x^2 f''(s) + 3x^2](1) + 3x^2(-1) = 3x^2 f''(s).\end{aligned}$$

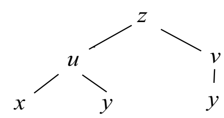


Thus,

$$\begin{aligned}\frac{\partial u}{\partial x} - 3x^2\frac{\partial u}{\partial y} + \frac{\partial^2 u}{\partial y \partial x} &= 3x^2[f'(s)+t+s] - 3x^2[f'(s)+t-s] + 3x^2 f''(s)\\ &= 3x^2 f''(s) + 6x^2 s\\ &= 3x^2[f''(s)+2s].\end{aligned}$$

3. Let $z = \frac{1}{2}v^2 - \frac{1}{2}u^2$, where $ux + ux^2 - y = 1$ and $vy + v^2y^2 = 2$. Find $\frac{\partial^2 z}{\partial y \partial x}$ and simplify your answer as much as possible.

From the schematic to the right,



$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} = -u\frac{\partial u}{\partial x}$

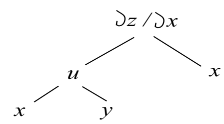
If we set $F(x, y, u) = ux + ux^2 - y - 1$, then

$$\frac{\partial u}{\partial x} = -\frac{\dfrac{\partial(F)}{\partial(x)}}{\dfrac{\partial(F)}{\partial(u)}} = -\frac{F_x}{F_u} = -\frac{u + 2ux}{x + x^2}.$$

Thus,

$$\frac{\partial z}{\partial x} = -u\left(-\frac{u + 2ux}{x + x^2}\right) = \frac{u^2(1 + 2x)}{x + x^2}.$$

From the schematic to the right,



$$\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial x}\right)\frac{\partial u}{\partial y} = \frac{2u(1 + 2x)}{x + x^2}\frac{\partial u}{\partial y}.$$

Since $\frac{\partial u}{\partial y} = -\frac{\dfrac{\partial(F)}{\partial(y)}}{\dfrac{\partial(F)}{\partial(u)}} = -\frac{F_y}{F_u} = -\frac{-1}{x + x^2} = \frac{1}{x + x^2}$,

we have

$$\frac{\partial^2 z}{\partial y \partial x} = \left[\frac{2u(1 + 2x)}{x + x^2}\right]\left(\frac{1}{x + x^2}\right) = \frac{2u(1 + 2x)}{(x + x^2)^2}.$$

An alterative way to find $\partial u/\partial x$ and $\partial u/\partial y$ is as follows:
Let $F(u, v, x, y) = ux + ux^2 - y - 1$ and $G(u, v, x, y) = vy + v^2y^2 - 2$. Then

$$\frac{\partial u}{\partial x} = -\frac{\dfrac{\partial(F,G)}{\partial(x,v)}}{\dfrac{\partial(F,G)}{\partial(u,v)}} = -\frac{\begin{vmatrix} F_x & F_v \\ G_x & G_v \end{vmatrix}}{\begin{vmatrix} F_u & F_v \\ G_u & G_v \end{vmatrix}} = -\frac{\begin{vmatrix} u + 2ux & 0 \\ 0 & y + 2vy^2 \end{vmatrix}}{\begin{vmatrix} x + x^2 & 0 \\ 0 & y + 2vy^2 \end{vmatrix}} = -\frac{(u + 2ux)(y + 2vy^2)}{(x + x^2)(y + 2vy^2)} = -\frac{u(1 + 2x)}{x + x^2},$$

$$\frac{\partial u}{\partial y} = -\frac{\dfrac{\partial(F,G)}{\partial(y,v)}}{\dfrac{\partial(F,G)}{\partial(u,v)}} = -\frac{\begin{vmatrix} F_y & F_v \\ G_y & G_v \end{vmatrix}}{\begin{vmatrix} F_u & F_v \\ G_u & G_v \end{vmatrix}} = -\frac{\begin{vmatrix} -1 & 0 \\ v + 2v^2y & y + 2vy^2 \end{vmatrix}}{\begin{vmatrix} x + x^2 & 0 \\ 0 & y + 2vy^2 \end{vmatrix}} = -\frac{-(y + 2vy^2)}{(x + x^2)(y + 2vy^2)} = \frac{1}{x + x^2}.$$

4. Find the number a in order that the tangent vectors to the curve

$$3x^2y + xyz + a = a + 4, \qquad xy^3 - xz + yz = 1$$

at the point $(1, 1, 1)$ have equal x and z components.

If we set $F(x, y, z) = 3x^2y + xyz + ax - a - 4$ and $G(x, y, z) = xy^3 - xz + yz - 1$, then

$$\nabla F_{|(1,1,1)} = (6xy + yz + a, 3x^2 + xz, xy)_{|(1,1,1)} = (7 + a, 4, 1),$$

and

$$\nabla G_{|(1,1,1)} = (y^3 - z, 3xy^2 + z, -x + y)_{|(1,1,1)} = (0, 4, 0).$$

A tangent vector to the curve at $(1, 1, 1)$ is

$$\begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 7+a & 4 & 1 \\ 0 & 4 & 0 \end{vmatrix} = (-4, 0, 28 + 4a).$$

x and z components are equal if $-4 = 28 + 4a$, from which $a = -8$.

5. Find the directional derivatives of the function $f(x, y, z) = xyz + z^2$ along the line

$$2x + y + z = 3, \qquad 3x - y + 2z = 8$$

at the point $(1, -1, 2)$.

$\nabla f_{|(1,-1,2)} = (yz, xz, xy + 2z)_{|(1,-1,2)} = (-2, 2, 3)$. A vector along the line is

$$\mathbf{v} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 2 & 1 & 1 \\ 3 & -1 & 2 \end{vmatrix} = (3, -1, -5).$$

The directional derivative of the function in this direction is

$$D_{\mathbf{v}}f = \nabla f \cdot \hat{\mathbf{v}} = (-2, 2, 3) \cdot \frac{(3, -1, -5)}{\sqrt{35}} = -\frac{23}{\sqrt{35}}.$$

The directional derivative in the direction $-\mathbf{v}$ is $23/\sqrt{35}$.

6. Find all critical points of the function

$$f(x,y) = y^3 + 3x^2y - 6x^2 - 6y^2 + 3,$$

and classify the critical point (x,y) for which $xy > 0$ as yielding a relative maximum, a relative minimum, a saddle point, or none of these.

For critical points, we solve

$$0 = f_x = 6xy - 12x = 6x(y-2), \qquad 0 = f_y = 3y^2 + 3x^2 - 12y.$$

From the first, either $x = 0$ or $y = 2$. When $x = 0$, the second equation requires $0 = 3y^2 - 12y = 3y(y-4)$. Thus, two critical points are $(0,0)$ and $(0,4)$. When $y = 2$, the second equation requires $0 = 3x^2 - 12$, from which $x = \pm 2$. Two more critical points are $(\pm 2, 2)$.

$$f_{xx} = 6(y-2), \qquad f_{xy} = 6x, \qquad f_{yy} = 6y - 12$$

At the critical point $(2,2)$, $B^2 - AC = (12)^2 - (0)(0) = 144 > 0$. Hence, the critical point $(2,2)$ yields a saddle point.