MATH2132 Test1 **October 4, 2018** **60 minutes**

Student Name - **Student Number -**

Values

5 **1.** Determine whether the sequence of functions

$$\left\{\frac{n^3x^2+nx-2}{2n^3x+5}\right\}, \quad -1 \le x \le 3,$$

has a limit as $n \to \infty$.

If we divide numerator and denominator by n^3,

$$\lim_{n\to\infty}\left(\frac{n^3x^2+nx-2}{2n^3x+5}\right) = \lim_{n\to\infty}\left(\frac{x^2+\dfrac{x}{n^2}-\dfrac{2}{n^3}}{2x+\dfrac{5}{n^3}}\right) = \frac{x^2}{2x} = \frac{x}{2},$$

provided $x \neq 0$. When $x = 0$, each function in the sequence is equal to $-2/5$, and therefore the limit at $x = 0$ is $-2/5$. Thus,

$$\lim_{n\to\infty}\left(\frac{n^3x^2+nx-2}{2n^3x+5}\right) = \begin{cases} x/2, & x \neq 0 \\ -2/5, & x = 0. \end{cases}$$

5 **2.** Determine whether the following series converges or diverges, and if it converges, find its sum:

$$\sum_{n=1}^{\infty}\frac{12^n}{n!}$$

Since $e^x = \displaystyle\sum_{n=0}^{\infty}\frac{x^n}{n!}$, we can say that

$$\sum_{n=1}^{\infty}\frac{12^n}{n!} = \sum_{n=0}^{\infty}\frac{12^n}{n!} - 1 = e^{12} - 1.$$

10 **3.** Find the interval of convergence for the power series

$$\sum_{n=12}^{\infty} \frac{(-1)^{n+1}n^2}{2^n}(x-2)^{2n}.$$

If we set $y = (x-2)^2$, the series becomes

$$\sum_{n=12}^{\infty} \frac{(-1)^{n+1}n^2}{2^n}y^n.$$

The radius of convergence of this series is

$$R_y = \lim_{n\to\infty} \left| \frac{\dfrac{(-1)^{n+1}n^2}{2^n}}{\dfrac{(-1)^{n+2}(n+1)^2}{2^{n+1}}} \right| = 2.$$

Thus, $R_x = \sqrt{2}$, and the open interval of convergence is

$$-\sqrt{2} < x-2 < \sqrt{2} \quad\Longrightarrow\quad 2-\sqrt{2} < x < 2+\sqrt{2}.$$

At the end points $x = 2 \pm \sqrt{2}$, the series becomes

$$\sum_{n=12}^{\infty} \frac{(-1)^{n+1}n^2}{2^n}(2 \pm \sqrt{2} - 2)^{2n} = \sum_{n=12}^{\infty} (-1)^{n+1}n^2.$$

Since $\lim\limits_{n\to\infty} (-1)^{n+1}n^2$, does not exist, the series diverges by the n^{th}-term test. The interval of convergence is $2-\sqrt{2} < x < 2+\sqrt{2}$.

8 **4.** Find the sum of the following series and its interval of convergence:

$$\sum_{n=2}^{\infty} \frac{2^{n+2}}{3^{n-1}}(x+1)^n$$

When we write the series in the form

$$\sum_{n=2}^{\infty} \frac{2^{n+2}}{3^{n-1}}(x+1)^n = \sum_{n=2}^{\infty} 12 \left[\frac{2}{3}(x+1)\right]^n,$$

we see that it is geometric with common ratio $(2/3)(x+1)$. Its sum is therefore

$$\frac{12\left[\frac{2}{3}(x+1)\right]^2}{1-\frac{2}{3}(x+1)} = \frac{16(x+1)^2}{1-2x}.$$

The interval of convergence is

$$-1 < \frac{2}{3}(x+1) < 1 \quad \Longrightarrow \quad -\frac{3}{2} < x+1 < \frac{3}{2} \quad \Longrightarrow \quad -\frac{5}{2} < x < \frac{1}{2}.$$

12 **5.** (a) Find the Taylor polynomial $P_2(x)$ for the function $\sin x$ about $x = 3\pi/4$.
(b) Use Taylor remainders to verify that the Taylor series for $\sin x$ about $x = 3\pi/4$ converges to $\sin x$ for all x.

(a)If we set $f(x) = \sin x$, then

$$f(3\pi/4) = \frac{1}{\sqrt{2}}, \qquad f'(3\pi/4) = \cos(3\pi/4) = -\frac{1}{\sqrt{2}}, \qquad f''(3\pi/4) = -\sin(3\pi/4) = -\frac{1}{\sqrt{2}}.$$

Thus,

$$\begin{aligned} P_2(x) &= f(3\pi/4) + f'(3\pi/4)(x - 3\pi/4) + \frac{f''(3\pi/4)}{2!}(x - 3\pi/4)^2 \\ &= \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\left(x - \frac{3\pi}{4}\right) - \frac{1}{2\sqrt{2}}\left(x - \frac{3\pi}{4}\right)^2. \end{aligned}$$

(b) Taylor remainders are

$$R_n(3\pi/4, x) = f^{(n+1)}(z_n)\frac{(x - 3\pi/4)^{n+1}}{(n+1)!}.$$

Now, $f^{(n+1)}(z_n)$ is one of the four quantities $\pm\cos z_n$ and $\pm\sin z_n$. Thus,

$$|R_n(3\pi/4, x)| \le (1)\frac{|x - 3\pi/4|^{n+1}}{(n+1)!},$$

and

$$\lim_{n\to\infty} |R_n(3\pi/4, x)| \le \lim_{n\to\infty} \frac{|x - 3\pi/4|^{n+1}}{(n+1)!} = 0.$$

Hence,

$$\lim_{n\to\infty} R_n(3\pi/4, x) = 0,$$

and the Taylor series for $\sin x$ about $x = 3\pi/4$ converges to $\sin x$ for all x.